Fishing for Complements

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(joint work with Lidia Angeleri Hügel and David Pauksztello)

§ 1. THE PROBLEM

Λ finite dimensional $\Bbbk$-algebra, $\operatorname{per}(\Lambda) = K^b(\operatorname{proj}\Lambda)$

Definition: $X \in \operatorname{per}\Lambda$ is: *presilting* if $\operatorname{Hom}_{\operatorname{per}(\Lambda)}(X, X[>0]) = 0$

Silting if $\begin{cases} \operatorname{Hom}_{\operatorname{per}(\Lambda)}(X, X[>0]) = 0 \\ \operatorname{thick}(X) = \operatorname{per}\Lambda \end{cases}$

A presilting $X \in \operatorname{per}\Lambda$ *admits a Complement* in $\operatorname{per}\Lambda$ if $\exists V \in \operatorname{per}\Lambda$ $X \oplus V$ silting.

Question: Does every presilting in $\operatorname{per}\Lambda$ admit a Complement in $\operatorname{per}(\Lambda)$?

Answer: [2023: LIU–ZHOU 2302.12502, JIN–SCHROLL–WANG 2303.17474, KALCK 2304.08417] NO!

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A presilting $X \in \mathrm{per}\,\Lambda$ *admits a complement* in $\mathrm{per}\,\Lambda$ if $\exists V \in \mathrm{per}\Lambda$ $X \oplus V$ silting.

Question: When Does ~~every~~ a presilting in $\mathrm{per}\,\Lambda$ admit a complement in $\mathrm{per}(\Lambda)$? or elsewhere ...?

§ 2. THE TOOLS

Large silting OBJECTS IN $D(\Lambda)$

- $\mathrm{Hom}\left(X, X^{(I)}[>0]\right) = 0, \ \forall I$
- $\mathrm{thick}\,(\mathrm{Add}\,X) = K^b(\mathrm{Proj}\,\Lambda)$

Co-t-structure $(\mathcal{U}, \mathcal{V})$

- $\mathrm{Hom}(\mathcal{U}, \mathcal{V}) = 0$
- $\mathcal{U}[-1] \subseteq \mathcal{U}$
- $\mathcal{U} * \mathcal{V} = \mathcal{T}$

THEOREM [Mendoza-Sáenz Santiago-Souto '13; Iyama-Yang '18; Nicolás-Saorín-Zvonareva '19; Angeleri Hügel-Marks-V '20]

For an object $Z \in D(\Lambda)$, denote by $\mathcal{V}_Z$ the subcategory $(Z[<0])^{\perp} = Z^{\perp_{>0}}$.

$\{\text{CO-T-STRUCTURES in } \mathrm{per}(\Lambda)\} \hookleftarrow \{\text{PRESILTING in } \mathrm{per}(\Lambda)\}/\sim \; \hookrightarrow \{\text{CO-T-STRUCTURES in } D(\Lambda)\}$

$({}^{\perp}\mathcal{V}_X, \mathcal{V}_X) \longleftarrow\!\shortmid X \longmapsto ({}^{\perp}\mathcal{V}_X, \mathcal{V}_X)$

${}^{\perp}\mathcal{V}_X = \mathrm{Cosusp}\ X[-1]$

$\{\text{BOUNDED CO-T-STRUCTURES in } \mathrm{per}(\Lambda)\} \xleftrightarrow{1:1} \{\text{SILTING in } \mathrm{per}(\Lambda)\}/\sim$

$\mathcal{V}_X = \mathrm{Susp}\ X$

$\{\text{LARGE SILTING in } D(\Lambda)\}/\sim \; \xleftrightarrow{1:1} \{\text{INTERMEDIATE CO-T-STRUCTURES in } D(\Lambda) \text{ WITH RIGHT ADJACENT T-STRUCTURE}\}$

$\to (X^{\perp_{>0}}, X^{\perp_{\leq 0}})$

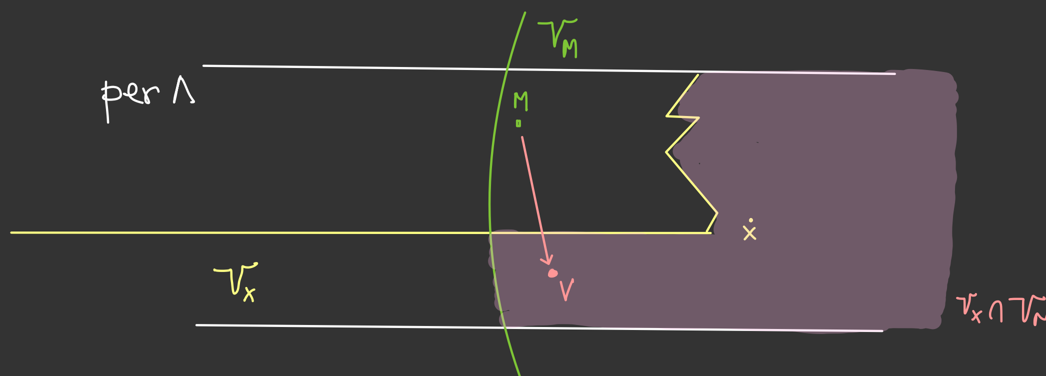
§ 3. THE THEOREMS

$\mathcal{V}_Z = (Z[<0])^{\perp} = Z^{\perp_{>0}}$

THEOREM 1 $X \in \operatorname{per}\Lambda$ presilting. The following are equivalent.

(1) X admits a complement;

(2) $\exists\, M \in \operatorname{per}\Lambda$ silting : $\begin{cases} X \in \mathcal{V}_M \\ ({}^{\perp}(\mathcal{V}_X \cap \mathcal{V}_M), \mathcal{V}_X \cap \mathcal{V}_M) \text{ co-t-structure in } \operatorname{per}\Lambda. \end{cases}$



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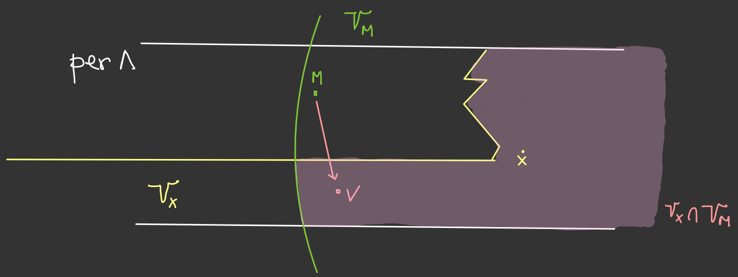
HOW TO FIND A COMPLEMENT?

$$\begin{array}{ccccccc} U & \longrightarrow & M & \longrightarrow & V & \longrightarrow & U[1] \\ \cap & & & & \cap & & \\ {}^{\perp}(\mathcal{V}_X \cap \mathcal{V}_M) & & & & \mathcal{V}_X \cap \mathcal{V}_M & & \end{array}$$

$\longrightarrow X \oplus V$ silting.



§ 3. The Theorems

$\mathcal{V}_Z = (Z[<0])^{\perp} = Z^{\perp_{>0}}$

Theorem 2 $X \in \operatorname{per} \Lambda$, $\operatorname{Hom}(X, X[>0]) = 0$. Then: $\exists\, V \in D(\Lambda)$: $X \oplus V$ large silting in $D(\Lambda)$

WHY?

[Saorín–Šťovíček '11] $\forall Z \in D(\Lambda)$, $({}^{\perp}\mathcal{V}_Z, \mathcal{V}_Z)$ co-t-structure in $D(\Lambda)$

- Take any (large) silting M with $X \in \mathcal{V}_M$ (e.g. $\Lambda[n]$ for some $n \ll 0$)
- $({}^{\perp}(\mathcal{V}_{X \oplus M}), \mathcal{V}_{X \oplus M} = \mathcal{V}_X \cap \mathcal{V}_M)$ is a co-t-structure in $D(\Lambda)$.
- $$\begin{array}{ccccccc} U & \longrightarrow & M & \longrightarrow & V & \longrightarrow & U[1] \\ \cap & & & & \cap & & \\ {}^{\perp}(\mathcal{V}_X \cap \mathcal{V}_M) & & & & \mathcal{V}_X \cap \mathcal{V}_M & & \end{array}$$

  $\rightsquigarrow X \oplus V$ is large silting.

! Construction of V hard to make explicit

Thank you for your attention