MEDLEY ON EXACT BOREL SUBALGEBRAS

Teresa Conde (Universität Bielefeld)

work in progress w/ Julian Külshammer
" " " w/ Steffen Koenig

$k \rightsquigarrow$ fixed field w/ $k = \bar{k}$

$A \rightsquigarrow$ algebra ("algebra" = fin. dim. k-algebra)

Thm [Wedderburn-Malcev] Any algebra has a maximal semi-simple subalgebra. Given two Morita equivalent algebras A and A' w/ basic maximal semi-simple subalgebras S, S' resp.

$\exists$ an isomorphism $f: A \to A'$ which restricts to an iso $f|_S : S \to S'$

Something similar happens for quasi-hereditary algs

$\Phi \rightsquigarrow$ isoclasses of simple A-modules

For $i \in \Phi$, $L_i \rightsquigarrow$ simple A-module w/ label i

$P_i \rightsquigarrow$ projective cover of L_i

$Q_i \rightsquigarrow$ injective hull of L_i

If $(\Phi, \leq)$ is a poset

$\Delta_i \rightsquigarrow$ largest quotient of P_i w/ composition factors L_j, $j \leq i$ (standard)

$\overline{\Delta}_i \rightsquigarrow$ " " X of Δ_i w/ $[X : L_i] = 1$ (proper standard)

$\nabla_i \rightsquigarrow$ " submodule of Q_i w/ composition factors L_j, $j \leq i$ (costandard)

$\overline{\nabla}_i \rightsquigarrow$ " " X of ∇_i w/ $[X : L_i] = 1$ (proper costandard)

Def The algebra A is left standardly stratified w.r.t. $(\Phi, \leq)$ if $\forall i \in \Phi$ $\mathrm{Ker}(P_i \twoheadrightarrow \Delta_i)$ is filtered by Δ_j, $j > i$. A is quasi-hereditary if $\Delta_i = \overline{\Delta}_i$, $\forall i \in \Phi$.

Def [Koenig'95] A subalgebra B of a left standardly stratified algebra $(A, \Phi, \leq)$ is an **exact Borel subalgebra** if:

(1) $A \otimes_B - : B\text{-Mod} \longrightarrow A\text{-Mod}$ is exact

(2) the simples over B are in bijection w/ Φ and $(B, \Phi, \leq)$ is left stand. strat. w/ simple standard modules

(3) $A \otimes_B L_i^B = \Delta_i^A$, $\forall i \in \Phi$.

B is **regular** if $\mathrm{Ext}_B^n(L_i^B, L_j^B) \cong \mathrm{Ext}_A^n(A\otimes_B L_i^B, A\otimes_B L_j^B)$, $\forall i,j \in \Phi, \forall n \geq 1$.

⚠ $\exists$ examples of left stand. strat. algs w/ no exact Borel subalgs

However, exact Borel subalgs exist up to equivalence ☺

Def Left stand. strat. algs $(A, \Phi, \leq)$ and $(A', \Phi', \leq')$ are equivalent if there exists an exact equivalence $\mathcal{F}(\Delta^A) \xrightarrow{\sim} \mathcal{F}(\Delta^{A'})$. cat of modules filtered by standards

Thm [Koenig-Külshammer-Ovsienko'14, Bautista-Pérez-Salmerón'23] Every left stand. strat. alg $(A, \Phi, \leq)$ is equivalent to another alg $(A', \Phi', \leq')$ which contains a ~~n~~ basic regular exact Borel subalgebra.

Thm [C'21] A' is unique up isomorphism.

Thm [Külshammer-Miemietz'21] Let $(A, \Phi, \leq)$ and $(A', \Phi', \leq')$ be two eqv qh algs with basic reg exact Borel subalgs, say B and B' resp.

$$\exists f: A \xrightarrow{\sim} A' \text{ s.t. } f|_B : B \xrightarrow{\sim} B'.$$

Which left stand. strat. algs contain a regular exact Borel subalg?

Given $(A, \Phi, \leq)$ left stand. strat.,

$\star\ [\Delta_i : L_j]$

$\star\ [\bar{\nabla}_i : L_j]$ $\quad \forall i,j \in \Phi$

$\star\ \dim(\mathrm{Hom}_A(\Delta_i, \Delta_j))$

$\xrightarrow{\text{INPUT}}$ **Algorithm** $\xrightarrow{\text{OUTPUT}}$ $\star$ Special matrix $V_{[(A,\Phi,\leq)]} = (v_{ij})_{i,j\in\Phi}$

Thm [C'21] Let $(A, \Phi, \leq)$ be left stand. strat..

(1) $(A, \Phi, \leq)$ has a regular exact Borel subalg iff the unique solution of $V_{[(A,\Phi,\leq)]}\, x = (\dim L_i^A)_{i\in\Phi}$ is a vector with entries in $\mathbb{N}$.

(2) If B is a regular exact Borel subalg of some alg eqv to A, then

$$\text{Cartan matrix of } B = \underbrace{\left(([\bar{\nabla}_i^A : L_j^A])_{i,j\in\Phi} \times V_{[(A,\Phi,\leq)]} \right)^T}_{=: CB_{[(A,\Phi,\leq)]}}$$

Easy criteria to decide whether an exact Borel subalg is reg?

Thm [C-Koenig] Let B be an exact Borel subalg of a basic left stand. strat. alg $(A, \Phi, \leq)$. TFAE

(a) B is regular

(b) $\operatorname{Rad} \Delta_i^A \in \mathcal{F}(\bar{\nabla}^A)$

(c) Cartan matrix of $B = C_{B_{[(A,\Phi,\leq)]}}$

What about compatibility of exact Borel subalgs under idempotent quotients/subalgs?

e idempotent in A induces a recollement

$$\begin{tikzcd}[column sep=huge] A/AeA\text{-Mod} \arrow[r, "\mathrm{inc}"] & A\text{-Mod} \arrow[r, "e(-)"] \arrow[l, bend right=50, "A/AeA \otimes_A -"'] \arrow[l, bend left=50, "\mathrm{Hom}_A(A/AeA, -)"] & eAe\text{-Mod} \arrow[l, bend right=50, "Ae \otimes_{eAe} -"'] \arrow[l, bend left=50, "\mathrm{Hom}_{eAe}(eA, -)"] \end{tikzcd}$$

Fix a left stand. strat. alg $(A, \Phi, \leq)$ and an idempotent $e \in A$.

Def The essential order $\leq_\varepsilon$ for $(A, \Phi, \leq)$ is the minimal partial order on Φ containing all pairs $(i,j) \in \Phi \times \Phi$ satisfying $[\Delta_j : L_i] \neq 0$ or $[\overline{\nabla}_j : L_i] \neq 0$.

Def The support of $e \in A$ is $\mathrm{supp}_A(e) := \{ i \in \Phi \mid e L_i^A \neq 0 \}$.

Def e is compatible with $(A, \Phi, \leq)$ if $\mathrm{supp}_A(e) \subseteq \Phi$ is upper closed subset.

Thm [C-Külshammer] Let B be a ~~n~~ Reg. exact Borel subalg of a left stand. strat. alg $(A, \Phi, \leq)$. Let $e \in B$ be an idempotent ~~supported in an upper closed subset of $(\Phi, \leq)$.~~ compatible w/ $(A, \Phi, \leq)$.

(1) eBe is a ~~n~~ Reg. exact Borel subalgebra of $(eAe, \mathrm{supp}(e), \leq)$

(2) B/BeB is a ~~n~~ Reg. " " " of $(A/AeA, \Phi \setminus \mathrm{supp}(e), \leq)$

Thm [C-Külshammer] Let $(A, \Phi, \leq)$ be a left stand. strat. alg. Let e be an idempotent compatible with $(A, \Phi, \leq)$. Then

$$V_{[(A, \Phi, \leq)]} = \left(\begin{array}{c|c} V_{[(A/AeA, \Phi \setminus \mathrm{supp}(e), \leq)]} & 0 \\ \hline * & V_{[(eAe, \mathrm{supp}(e), \leq)]} \end{array}\right)$$

Thank You For Listening!