Maurer-Cartan equation for gentle algebras

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joint work with M. Müller, M. J. Redondo, and P. Suarez.

Flash Talks in Representation Theory V

$\operatorname{char}\mathbb{k} = 0$, all algebras are finite-dimensional $\mathbb{k}$-algebras

Motivation:

Deformations		Maurer-Cartan Elements
$A_f = A \otimes_{\mathbb{k}} \mathbb{k}[t]$		
$f : A \otimes A \to A_f$		$f_i \in C^2(A)$
$f(a,b) = f_0(a,b) + \sum_{i \geq 1} f_i(a,b)t^i$	$\rightsquigarrow$	$df_k + \frac{1}{2}\sum_{i+j=k}[f_i, f_j] = 0$
associative		

Deformations

Let $A_f := A \otimes_{\Bbbk} \Bbbk[t] \simeq A[t]$.
An element $\alpha \in A[t]$ is $\alpha = a_0 + a_1 t + \cdots + a_n t^n + \ldots$.
Let $f_i \in \mathrm{Hom}_{\Bbbk}(A \otimes A, A)$ and

$$f(a, b) = \sum f_i(a, b) t^i.$$

The product $f : A[t] \otimes A[t] \to A[t]$ is given by

$$f(\alpha, \beta) = \sum_{i,j} f(a_i, b_j) t^{i+j}.$$

If f_0 is the original product of A and f is associative, then A_f is a *deformation* of A.

A L_∞-algebra is a vector space L:

- $\mathbb{Z}$-graded: $L = \oplus_{n\in\mathbb{Z}} L^n$;
- $l_n : \otimes^n L \to L$ linear of degree $2-n$ such that

$$l_n\ \hat{\sigma} = \chi(\sigma)\ l_n \text{ and}$$
$$\sum_{i+j=n+1} \sum_{\sigma\in\mathbb{S}_{i,n-i}} (-1)^{i(j-1)}\chi(\sigma)\ l_j(l_i \otimes Id^{\otimes n-i})\ \hat{\sigma} = 0.$$

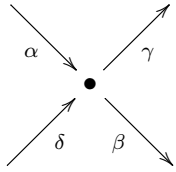
Maurer-Cartan elements: $f \in L^1$ such that

$$l_1(f) + \sum_{n>1} \frac{1}{n!} l_n(f,\dots,f) = 0.$$

Gentle algebras

A quadratic algebra $A = \Bbbk Q/I$ is gentle if:

1. Q is finite;
2. For each $\alpha \in Q_1$, there exists at most one arrow β with $s(\beta) = t(\alpha)$ such that $\alpha\beta \in I$, and at most one arrow γ with $t(\gamma) = s(\alpha)$ such that $\gamma\alpha \in I$. These relations generate I;
3. For each $\alpha \in Q_1$, there exists at most one arrow β such that $\alpha\beta \notin I$, and at most one arrow γ such that $\gamma\alpha \notin I$.



Barzdell's complex

$$B^*(A) = (\mathrm{Hom}_{E-E}(\Bbbk AP_n, A), (-1)^n \delta^n)_{n\geq 0}$$

where AP_n is the set of supports of n-concatenations associated to a presentation (Q, I) of A, $E = \Bbbk Q_0$. $B^*(A)[1]$ is a L_∞-algebra.

To find MC elements we only need $f \in B^1(A)[1] = B^2(A)$.
For them we should calculate

$$l_n(f_1, \ldots, f_n)(\alpha_1, \alpha_2, \alpha_3)$$

where $(\alpha_1, \alpha_2, \alpha_3) \in AP_3$, to find solutions of the generalized MC equations

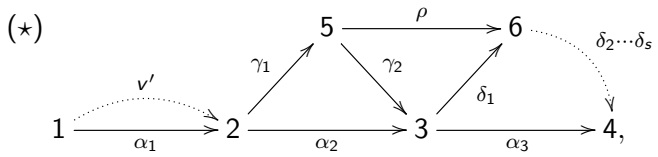
$$-\delta^2(f_i) - \sum_{n\geq 2} \sum_{j_1+\cdots+j_n=i} (-1)^{\frac{(n+1)n}{2}} \frac{1}{n!} l_n(f_{j_1}, \cdots, f_{j_n}) = 0.$$

Lemma ($l_2 = 0$)
Let $A = \Bbbk Q/I$ be a gentle algebra and $\omega = (\alpha_1, \alpha_2, \alpha_3) \in AP_3$. Then $l_2(f_1, f_2)(\omega) = 0$ for all $f_1, f_2 \in B^1(A)[1]$.

Lemma ($l_3 = 0$)
Let $A = \Bbbk Q/I$ be a gentle algebra and $\omega = (\alpha_1, \alpha_2, \alpha_3) \in AP_3$. If Q has no oriented cycles and no parallel arrows, then $l_3(f_1, f_2, f_3)(\omega) = 0$ for all $f_1, f_2, f_3 \in B^1(A)[1]$.

Proposition (no nilpotent)

Let $A = kQ/I$ be a gentle algebra with quiver



where $\omega = (\alpha_1, \alpha_2, \alpha_3) \in AP_3$, $v', \gamma_1\gamma_2$ and $w' = \delta_1 \cdots \delta_s$ are nonzero paths. Then, there exists $f \in B^1(A)[1]$ such that $l_n(f, \ldots, f)(\omega) \neq 0$ for $n = 2 + 3k, k \geq 1$.

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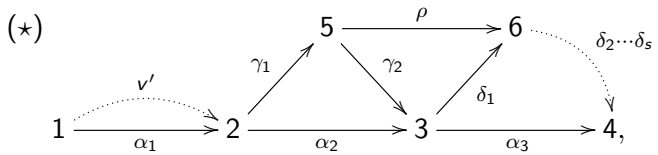

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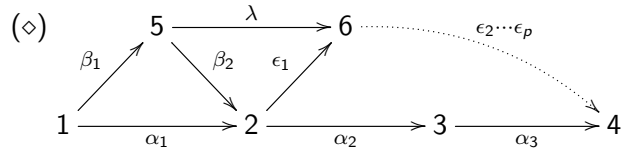
Theorem (nilpotent) Let $A = \Bbbk Q/I$ be a gentle algebra. Assume Q has no oriented cycles and no parallel arrows. If Q does not contain a subquiver of the form $(\star)$, then $l_n(f_1, \ldots, f_n) = 0$ for all $n \geq 5$ and $f_i \in B^1(A)[1], 1 \leq i \leq n$.

Theorem ($l_4 \neq 0$) Let $A = \Bbbk Q/I$ be a gentle algebra. Assume Q has no oriented cycles and no parallel arrows. If $l_4(f_1, f_2, f_3, f_4) \neq 0$ for some $f_i \in B^1(A)[1]$, then Q contains a subquiver of the form



with $\omega = (\alpha_1, \alpha_2, \alpha_3) \in AP_3$, $\beta_1\beta_2 \neq 0$, $\epsilon_1 \cdots \epsilon_p \neq 0$, and $l_4(f_1, f_2, f_3, f_4)(\omega) \neq 0$ for some $f_i \in B^1(A)[1],, 1 \leq i \leq 4$.

Theorem (MC elements) Let $A = \Bbbk Q/I$ be a gentle algebra. Assume Q has no oriented cycles, no parallel arrows, and no subquivers of the form $(\star)$. Then $f = \sum_{i=1}^{m} f_i t^i$ is a MC element if and only if f_i is a 2-cocycle for all i.

Muchas gracias!!

Examples

$$Q: \qquad 1 \xrightarrow[\alpha_1]{} 2 \underset{\alpha_2}{\overset{\beta}{\rightrightarrows}} 3 \underset{\alpha_3}{\overset{\gamma}{\rightrightarrows}} 4$$

and the ideal $I =< \alpha_1\alpha_2, \alpha_2\alpha_3, \beta\gamma >$.

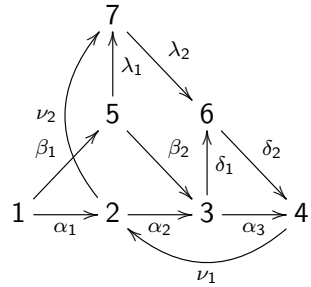
Let $A = \Bbbk Q/I$ and

$$f = (\alpha_1\alpha_2||\alpha_1\beta) + (\alpha_2\alpha_3||\alpha_2\gamma) + (\beta\gamma||\alpha_2\gamma + \beta\alpha_3) \in B^2(A).$$

Then

$$l_n(f, \ldots, f)(\alpha_1, \alpha_2, \alpha_3) = \begin{cases} (-1)^{\frac{n-1}{2}} n!\ \alpha_1\beta\alpha_3, & n \text{ odd;} \\ 0, & n \text{ even.} \end{cases}$$

Q :



and the ideal $I =< \alpha_1\alpha_2, \alpha_2\alpha_3, \nu_1\nu_2, \beta_2\delta_1, \beta_1\lambda_1, \delta_2\nu_1, \nu_2\lambda_2, \lambda_2\delta_2 >$. Let $A = \Bbbk Q/I$ and

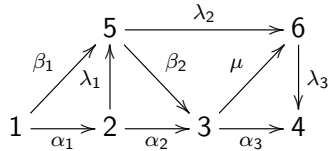
$$f = (\alpha_1\alpha_2||\beta_1\beta_2 + \beta_1\beta_2\alpha_3\nu_1\alpha_2) + (\alpha_2\alpha_3||\alpha_2\delta_1\delta_2) + (\beta_2\delta_1||\beta_2\alpha_3\nu_1\alpha_2\delta_1 + \lambda_1\lambda_2) + (\beta_1\lambda_1||\alpha_1\nu_2) + (\nu_2\lambda_2||\alpha_2\delta_1).$$

Then

$$l_n(f, \ldots, f)(\alpha_1, \alpha_2, \alpha_3) = \begin{cases} -n!\beta_1\beta_2\alpha_3\nu_1\alpha_2\delta_1\delta_2, & \text{if 3 divides } n; \\ 0, & \text{otherwise.} \end{cases}$$

More examples

Let $A = \Bbbk Q/I$ be the string algebra given by the quiver



with $I = < \alpha_1\alpha_2, \alpha_2\alpha_3, \lambda_1\beta_2, \beta_1\lambda_2, \beta_2\alpha_3, \alpha_2\mu, \mu\lambda_3 >$. Set

$$f = (\alpha_1\alpha_2||\beta_1\beta_2) + (\beta_2\alpha_3||\lambda_2\lambda_3) + (\beta_1\lambda_2||\beta_1\beta_2\mu + \alpha_1\lambda_1\lambda_2) + (\mu\lambda_3||\alpha_3).$$

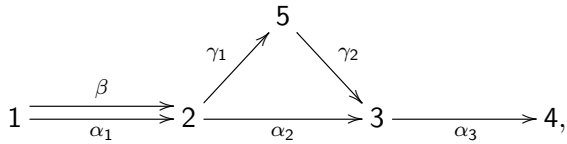
Then, one can check that

$$l_{3k}(f, \ldots, f)(\alpha_1, \alpha_2, \alpha_3) = c_k\, \alpha_1\lambda_1\lambda_2\lambda_3 \neq 0$$

for some nonzero $c_k \in \Bbbk$.

More examples

Let $A = \Bbbk Q/I$ be the algebra with quiver



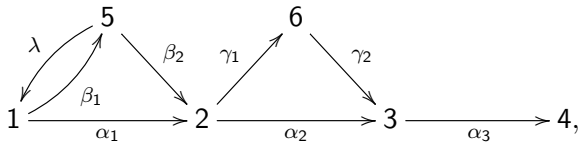
and $I = \langle \alpha_1\alpha_2, \alpha_2\alpha_3, \beta\gamma_1 \rangle$. Let
$f_1 = (\alpha_1\alpha_2||\beta\alpha_2 + \alpha_1\gamma_1\gamma_2) + (\alpha_2\alpha_3||\gamma_1\gamma_2\alpha_3) + (\beta\gamma_1||\alpha_1\gamma_1)$ and $f_3 = (\alpha_1\alpha_2||\alpha_1\gamma_1\gamma_2)$.
One can check that $l_n(g_1, \ldots, g_n)(\alpha_1, \alpha_2, \alpha_3) = 0$ for all $g_j \in B^1(A)[1]$ and for all $n \neq 1, 3$. Moreover, f_1 is a 2-cocycle and $l_3(f_{i_1}, f_{i_2}, f_{i_3})(\alpha_1, \alpha_2, \alpha_3) = 0$ except for

$$l_3(f_1, f_1, f_1)(\alpha_1, \alpha_2, \alpha_3) = 6\alpha_1\gamma_1\gamma_2\alpha_3 \neq 0.$$

As $l_1(f_3)(\alpha_1, \alpha_2, \alpha_3) = \alpha_1\gamma_1\gamma_2\alpha_3 \neq 0$ we have that f_3 is not a 2-cocycle, and $l_1(f_3) - \frac{1}{6} l_3(f_1, f_1, f_1) = 0$, hence $f = f_1 t + f_3 t^3$ is a Maurer-Cartan element.

More examples

Let $A = \Bbbk Q/I$ be the algebra with quiver



and $I = \langle \alpha_1\alpha_2, \alpha_2\alpha_3, \beta_2\gamma_1, \beta_1\lambda, \lambda\beta_1 \rangle$. Let
$f_1 = (\alpha_1\alpha_2||\beta_1\beta_2\alpha_2 + \alpha_1\gamma_1\gamma_2) + (\alpha_2\alpha_3||\gamma_1\gamma_2\alpha_3) + (\beta_2\gamma_1||\lambda\alpha_1\gamma_1) + (\beta_1\lambda||e_1) + (\lambda\beta_1||e_5)$
and $f_4 = -(\alpha_1\alpha_2||\alpha_1\gamma_1\gamma_2)$. One can check that $l_n(g_1, \ldots, g_n)(\omega) = 0$ for all $g_j \in B^1(A)[1]$, for all $\omega \in AP_3$, and for all $n \neq 1, 4$. Moreover, f_1 is a 2-cocycle and $l_4(f_{i_1}, f_{i_2}, f_{i_3}, f_{i_4})(\omega) = 0$ except for

$$l_4(f_1, f_1, f_1, f_1)(\alpha_1, \alpha_2, \alpha_3) = -24\alpha_1\gamma_1\gamma_2\alpha_3 \neq 0.$$

Since $l_1(f_4)(\alpha_1, \alpha_2, \alpha_3) = -\alpha_1\gamma_1\gamma_2\alpha_3 \neq 0$ we have that f_4 is not a 2-cocycle and $f = f_1 t + f_4 t^4$ is a Maurer-Cartan element because $l_1(f_4) - \frac{1}{24} l_4(f_1, f_1, f_1, f_1) = 0$.