An introduction to geometric mechanics

Brynjulf Owren

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Lagrangian mechanics, variational calculus. We here denote by Q a differentiable d-dimensional manifold of generalized positions, and its tangent bundle TQ represents the velocity phase space. We introduce coordinates on Q, $(q_1, \ldots, q_d)$ which induces coordinates $(q_1, \ldots, q_d, v_1, \ldots, v_d)$ on TQ. The Lagrangian L is a function on TQ which we may express in coordinates

$$L : TQ \to \mathbb{R}, \qquad L(q_1, \ldots, q_d, v_1, \ldots, v_d)$$

For any differentiable curve $q(t) \in Q$ and real numbers $t_1 < t_2$ we define the *action*

$$S = \int_{t_1}^{t_2} L(q(t), \dot{q}(t))\, \mathrm{d}t \tag{1}$$

Our aim is to find the choice of $q(t)$ with fixed endpoints $q(t_1), q(t_2)$ which minimizes S. But we take a somewhat weaker approach, we look for a stationary path $q(t)$. The approach we use is to consider all possible small variations $\delta q(t)$ of a candidate curve $q(t)$ and show that they would all increase (or decrease) the value of S. We assume $\delta(t_1) = \delta(t_2) = 0$ to keep the end points fixed under the variation.

$$\delta S = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \int_{t_1}^{t_2} L(q_\varepsilon(t), \dot{q}_\varepsilon(t))\, \mathrm{d}t = 0$$

The curve $q_\varepsilon(t)$ is such that

$$q_\varepsilon(t)|_{\varepsilon=0} = q(t), \qquad \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} q_\varepsilon(t) = \delta q(t) \in T_{q(t)}Q$$

Assuming we can take the derivative inside the integral, we get

$$\delta S = \int_{t_1}^{t_2} \left(\langle \frac{\partial L}{\partial q}(q, \dot{q}), \delta q \rangle + \langle \frac{\partial L}{\partial \dot{q}}(q, \dot{q}), \delta \dot{q} \rangle \right) \mathrm{d}t$$

We now do integration by parts on the second term in the integral,

$$\delta S = \int_{t_1}^{t_2} \left\langle \frac{\partial L}{\partial q}(q, \dot{q}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}(q, \dot{q}) \right), \delta q \right\rangle \mathrm{d}t + \left[\frac{\partial L}{\partial \dot{q}}(q, \dot{q}) \delta q \right]_{t_1}^{t_2}$$

Hamilton's principle states that the path chosen by a dynamical system is a stationary path for the action integral (1). Using the boundary conditions on δq, and assuming that L is continuously differentiable with respect to both arguments, we conclude that $\delta S = 0$ for all δq requires

$$\frac{\partial L}{\partial q}(q,\dot{q}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}(q,\dot{q})\right) = 0$$

These are the *Euler-Lagrange equations.*

Exercise 1 In the last step before arriving at the Euler-Lagrange equations, we made use of the following result. If $f(t)$ is a continuous function on $[a,b]$, then $\int_a^b f(t)h(t)\,\mathrm{d}t = 0$ for all continuous $h(t)$ requires $f(t) \equiv 0$ on $[a,b]$ (in fact it is even enough to assume that the integral vanishes for all $h \in C^\infty[a,b]$). Prove this result. □

Example 1 Naturally, the Lagrangian L is the difference between the kinetic and potential energy of the system. We consider the dynamics of d particles

$$L(q,\dot{q}) = \frac{1}{2}\sum_i m_i|\dot{q}_i|^2 - U(q)$$

Here, $q_i \in \mathbb{R}^3$ is the position coordinates of the ith particle, and its velocity is $\dot{q}_i$. So $q = (q_1,\ldots,q_d) \in \mathbb{R}^{3d}$. The potential energy $U(q)$ is assumed to only depend on the positions q. We compute

$$\frac{\partial L}{\partial q_i} = -\frac{\partial U}{\partial q_i}, \qquad \frac{\partial L}{\partial \dot{q}_i} = m_i\dot{q}_i$$

so that the Euler–Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_i\dot{q}_i) = -\frac{\partial U}{\partial q_i}$$

This is nothing else than Newton's second law, mass times acceleration on the left, and a conservative force field on the right.

□

The Legendre transformation. As discussed previously, the partial derivatives of L both with respect to q and $\dot{q}$ are dual vectors, i.e. elements of T_q^*Q. Let

$$p = \frac{\partial L}{\partial \dot{q}}(q,\dot{q}), \tag{2}$$

this is called the *Legendre transformation*, the map $(q,\dot{q}) \mapsto (q,p)$ maps TQ to T^*Q. The Legendre transform is invertible if its hessian matrix

$$\frac{\partial^2 L}{\partial \dot{q}^2}(q,\dot{q})$$

is non-singular. The Hamiltonian function $H(q,p)$ is obtained as

$$H(q,p) = \langle p,\dot{q}\rangle - L(q,\dot{q}) \tag{3}$$

where it is understood that (2) is solved for $\dot{q}$ in terms of (q, p) and inserted into (3). Now the differential equations for q and p can be found from

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q} \tag{4}$$

Exercise 2 For the Lagrangian in Example 1, determine the Legendre transformation and the resulting Hamiltonian function in terms of q and p. □

Symplectic manifolds and Hamiltonian systems A differentiable manifold M together with a differential two-form ω is a symplectic manifold if ω is non-degenerate and closed.

1. Non-degenerate. This means that for all $m \in M$ and non-zero $u \in T_m M$, there is a $v \in T_m M$ such that $\omega(u, v) \neq 0$. Another way to think of non-degeneracy is in terms of matrices. Suppose that $e_1, \ldots, e_n$ is a basis for $T_m M$ and form the skew-symmetric matrix S with entries $S_{ij} = \omega(e_i, e_j)$. Then non-degeneracy of ω is equivalent to S being non-singular.

2. Closed. This means that $\mathrm{d}\omega = 0$. Note that all exact two-forms are closed, these are forms that can be written as the differentials of a one-form, i.e. $\omega = \mathrm{d}\theta$. In star-shaped domains, such as $\mathbb{R}^n$, all closed form are exact.

To every vector field X on a manifold, there corresponds a map i_X that takes k-forms to $k-1$-forms defined as

$$(\mathrm{i}_X \omega)(v_1, \ldots, v_{k-1}) = \omega(X, v_1, \ldots, v_{k-1})$$

This map is called an interior derivative.

Now let H be any smooth function on M. Then X_H is the corresponding Hamiltonian vector field for the symplectic two-form ω if

$$\mathrm{dH} = \mathrm{i}_{X_H} \omega \tag{5}$$

Example 2 The most natural symplectic manifold is the cotangent bundle of some other manifold, i.e. $M = T^*Q$. This manifold has dimension $n = 2 \dim Q = 2d$. In local coordinates $(q_1, \ldots, q_d, p_1, \ldots, p_d)$, the symplectic form is

$$\omega = \mathrm{d}p_1 \wedge \mathrm{d}q_1 + \cdots + \mathrm{d}p_d \wedge \mathrm{d}q_d \tag{6}$$

The corresponding basis for the tangent space is $(e_{q_1}, \ldots, e_{q_d}, e_{p_1}, e_{p_d})$, meaning that

$$\mathrm{d}q_i(e_{q_j}) = \delta_{i,j}, \quad \mathrm{d}p_i(e_{p_j}) = \delta_{i,j}, \quad \mathrm{d}q_i(e_{p_j}) = 0, \quad \mathrm{d}p_i(e_{q_j}) = 0$$

With this you can verify that the matrix[1] of the form ω is

$$J = \begin{bmatrix} 0 & -I_{d\times d} \\ I_{d\times d} & 0 \end{bmatrix}$$

[1] In this example we do not worry about mixing together vectors and covectors in the notation we use, in calculating things in coordinates it is however common to distinguish these two by letting one kind be column vectors and the other kind be row vectors

Let us check if the general abstract definition works in this case. We look at

$$\mathrm{i}_X\omega = \omega(X,\cdot) = JX = \mathrm{d}H = (\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p})^T$$

which leads precisely to (4). □

Symplectic maps. A map $\Psi : M \to M$ on a symplectic manifold is one which preserves the symplectic two-form. Mathematically, this just means that for any point $m \in M$, and two tangent vectors u, v in $T_m M$, one has

$$\omega(T_m\Psi u, T_m\Psi v) = \omega(u, v)$$

The expression on the left hand side is called the pullback of the form ω one used the notation $\Psi^*\omega$, so that

$$(\Psi^*\omega)(u, v) = \omega(T_m\Psi u, T_m\Psi v)$$

Then to express preservation of ω we simply write $\Psi^*\omega = \omega$. Whenever the map Ψ is the flow φ_t of a vector field X, we can use the Lie derivative of ω

$$\mathcal{L}_X(\omega) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi_t^*\omega,$$

so the flow of X preserves ω if and only if $\mathcal{L}_X = 0$. There is a very important and beautiful formula for the Lie derivative (of any differential form) called *Cartan's Magic Formula*, see e.g. [2]

$$\mathcal{L}_X\omega = \mathrm{d}\mathrm{i}_X\omega + \mathrm{i}_X\mathrm{d}\omega \tag{7}$$

The verification this formula involves straightforward, but lengthy algebraic manipulations and will therefore be omitted here.

Example 3 We consider again the canonical symplectic form (6) on T^*Q. The tangent map is then nothing else than the Jacobian matrix

$$T_m\Psi = \frac{\partial \Psi}{\partial(q,p)} = \frac{\partial \psi}{\partial y}$$

where we write $y = (q, p) \in \mathbb{R}^{2d}$. So for the preservation of ω we get

$$\left(\frac{\partial \Psi}{\partial y} u\right)^T J \frac{\partial \Psi}{\partial y} v = u^T J v, \quad \forall u, v$$

which is equivalent to

$$\left(\frac{\partial \Psi}{\partial y}\right)^T J \frac{\partial \Psi}{\partial y} = J$$

a condition you encounter frequently for instance in [1]. □

Flows of Hamiltonian systems are symplectic maps. Suppose that we have a Hamiltonian H on a symplectic manifold. The vector field X_H resulting from (5) can be inserted into (7)

$$\mathcal{L}_{X_H}\omega = \mathrm{d}\mathrm{i}_{X_H}\omega + \mathrm{i}_{X_H}\mathrm{d}\omega$$

Now, the second term on the right hand side vanishes because ω is closed, whereas the first term is, according to, (5) equal to $\mathrm{d}(\mathrm{d}H) = 0$. So we conclude that $\mathcal{L}_{X_H}\omega = 0$ which we have seen is equivalent to φ_t being symplectic.

Conversely, suppose that we had started from a vector field X whose flow is symplectic, could we then conclude that X is a Hamiltonian vector field? The answer follows again from Cartan's Magic Formula, the condition $\mathcal{L}_X\omega = 0$ would then be equivalent to

$$\mathrm{d}\mathrm{i}_X\omega = 0$$

so the condition would be that $i_X\omega$ is a closed one-form. Vector fields with this property are called *locally Hamiltonian* since a closed form will be exact on some open set around any point m, but not necessarily in a global sense.

References

[1] Ernst Hairer, Christian Lubich, and Gerhard Wanner. *Geometric numerical integration*, volume 31 of *Springer Series in Computational Mathematics*. Springer, Heidelberg, 2010. Structure-preserving algorithms for ordinary differential equations, Reprint of the second (2006) edition.

[2] J. E. Marsden and T. S. Ratiu. *Introduction to mechanics and symmetry*, volume 17 of *Texts in Applied Mathematics*. Springer-Verlag, New York, second edition, 1999. A basic exposition of classical mechanical systems.