Preservation of first integrals.

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Structure-preserving discretization of differential equations

Overview

- Summary of what we have seen so far on preservation of first integrals. linear and quadratic invariants
- No RK-methods preserve all **cubic invariants**: counterexample.
- **Discrete gradients** and discrete gradient methods.
- Dissipation of energy.
- Preservation of energy with RK-methods for Hamiltonian vector fields with a polynomial Hamiltonian function
- Hamiltonian PDEs?

The lecture is based on the note published on the webpage of the course under *lecture 6* and in particular section 1, 2, 3.

$$\dot{y} = f(y), \tag{1}$$

with $y(t) \in \mathbf{R}^m$ for all t.

Definition

A function $\mathcal{I} : \mathbf{R}^m \to \mathbf{R}$ is a first integral (or invariant) of the ODE if and only if

$$\nabla \mathcal{I}(y)^T f(y) = 0, \quad \forall y. \tag{2}$$

Linear invariants: $\mathcal{I}(y) := d^T y$

Theorem

All Runge-Kutta methods preserve all linear invariants.

Quadratic invariants: $\mathcal{I}(y) := y^T Cy$ where C is a symmetric $m \times m$ matrix.

Theorem

Runge-Kutta methods satisfying

$$b_i a_{i,j} + b_j a_{j,i} = b_i b_j, \quad \forall i, j = 1, \ldots, s, \tag{3}$$

preserves all quadratic invariants.

Cubic invariants ?

Quadratic invariants: $\mathcal{I}(y) := y^T C y$ where C is a symmetric $m \times m$ matrix.

Theorem

Runge-Kutta methods satisfying

$$b_i a_{i,j} + b_j a_{j,i} = b_i b_j, \quad \forall i, j = 1, \ldots, s, \tag{3}$$

preserves all quadratic invariants.

Cubic invariants ?

Theorem

No Runge-Kutta methods preserve all polynomial invariants of degree 3 or higher.

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