Structure-preserving discretization of differential equations, AARMS summer school 2015

July 9, 2015

FIRST ASSIGNMENT

In this first assignment we want to implement some Runge-Kutta methods, verify numerically their order of convergence and see how well they preserve the invariants of the considered test equations.

In this assignments we consider the pendulum equations written as a Hamiltonian system with Hamiltonian $H(q,p) = \frac{1}{2}p^2 - \cos(q)$,

$$\dot{q} = p, \qquad (1)$$

$$\dot{p} = -\sin(q) \qquad (2)$$

with $q(0) = 2$ and $p(0) = 0$.
We will use the following Runge-Kutta methods.
The forward Euler method:

$$y_{n+1} = y_n + h\, f(y_n).$$

The improved Euler method:

$$y_{n+1} = y_n + \frac{h}{2}\big(f(y_n) + f(y_n + hf(y_n))\big).$$

The midpoint rule:

$$y_{n+1} = y_n + h\, f\big(\frac{y_n + y_{n+1}}{2}\big).$$

a) Implement the forward Euler method, the improved Euler method and the mid-point rule to solve the pendulum equations. Integrate on the time interval $[0,1]$ and consider different values of the step size: $h = 1/N$ and $N = N_0 \cdot 2^n$, $n = 0,\ldots,4$. Take e.g. $N_0 = 5$. Compute a reference solution ("exact solution") using the `ode45` built-in routine of MATLAB with absolute tolerance $1e-11$ and relative tolerance $1e-10$, and then compute the error $\|e_N\|_2 = \|y(1) - y_N\|_2$ for the various values of h. Provide error plots showing that the methods have the expected order of convergence, i.e. order 1 for the forward Euler and order 2 for the other two methods. Use `loglog` in MATLAB[1] and plot the step sizes versus the corresponding norms of the errors.

[1] The function `loglog` in MATLAB plots the data in a logarithmic scale. This way plotting the error as a function of h should result in a straight line with slope 1 for the forward Euler and 2 for the other two methods.

You should hand in one plot showing that you obtained the correct order for the methods.

b) Use the midpoint method and the improved Euler method to solve the pendulum equations. Solve the problem on the interval $[0, 4\pi]$. First, integrate numerically with 200 steps, then reduce the number of steps to 20. Looking at the phase plot (i.e. the plot of q vs p), you should see a substantial difference in the behaviour of the improved Euler method with the two different step sizes. Next use the time interval $[0, 4000]$ with step size $h = 0.5$. Also with this large time interval, you can verify that the midpoint rule produces closed orbits in the phase space, while this is not the case for the improved Euler method. Compute and plot the error in the Hamiltonian for the two methods.

You should hand in a phase space plot showing the different behaviour for the methods (the experiment with 20 steps), and one plot showing the error in the energy for the two methods as a function of time.

Send your answers (files) by e-mail to elenac@math.ntnu.no by Wednesday July 15th, 15:00 pm.