FINAL EXAM FOR THE AARMS/PIMS SUMMER SCHOOL
Structure-preserving discretization of differential equations
Halifax, Nova Scotia, July 30, 2015

Instructions and guidelines

- This is a take-home exam, it is expected that you do the problems without help of any other person
- You can use any written material to aid you in solving the problems, including the course material
- The exam consists of two parts. The first part is a google form that you reach via a link on the course home page. The second part consists of Problem 1-3 below
- You can start doing the exam immediately after it is available on the course web page
- The deadline for submission of the two parts of the exam is Monday, August 3, 10pm (Halifax time). The answers to part 2 should be sent by email to Elena Celledoni (`elenac@math.ntnu.no`), with your name clearly indicated on the document. pdf-format is preferred.

Problem 1

a) Find the adjoint of the Runge-Kutta method

$$y_{n+1} = y_n + h\left[-\frac{1}{2}f(y_n) + 2f\big(\frac{y_n + y_{n+1}}{2}\big) - \frac{1}{2}f(y_{n+1})\right]. \qquad (1)$$

b) Explain why the midpoint method applied to linear Hamiltonian vector fields is both energy preserving and symplectic.

c) Show that the method (1) is symplectic and energy-preserving when applied to linear Hamiltonian vector fields. Does any of these two properties hold true when the Hamiltonian vector field is quadratic?

Problem 2 We are given a system of ordinary differential equations

$$\begin{aligned}\dot{x} &= 2xy - x^2\\ \dot{y} &= 2yz - y^2\\ \dot{z} &= 2xz - z^2\end{aligned} \qquad (2)$$

a) Show that the system is divergence free and use the method of Feng and Shang to write down the corresponding two Hamiltonian vector fields $f_{1,2}$ and $f_{2,3}$ as well as the matching Hamiltonians $H_{1,2}$ and $H_{2,3}$.

b) Write down a a consistent volume preserving integrator for this problem.

Problem 3 If we apply the method of Kahan (1) to the problem

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -x^2\end{aligned}$$

it results in the method map

$$\Phi_h \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x+hy}{1+\frac{1}{2}h^2x} \\ \frac{y-(hx^2+\frac{1}{2}h^2xy)}{1+\frac{1}{2}h^2x} \end{pmatrix}$$

which has the following expansion in terms of the stepsize h

$$\Phi_h \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + h \begin{pmatrix} y \\ -x^2 \end{pmatrix} + h^2 \begin{pmatrix} -\frac{1}{2}x^2 \\ -xy \end{pmatrix} + h^3 \begin{pmatrix} -\frac{1}{2}xy \\ \frac{1}{2}x^3 \end{pmatrix} + \cdots$$

Find the coefficient functions f_2 and f_3 in the expansion for the modified vector field.