Regarding discrete variational deivatives and discrete gradients

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Introduction

In the assignment 2 it is easy to misunderstand the procedure so that the resulting discretization becomes non consistent (one easily ends up with is a misplaced factor Δx in the method). The right way to proceed is to seek for $\bar{H}$ such that $\Delta x \bar{H} \approx \mathcal{H}$.

Further explanation

We are solving numerically

$$u_t = \mathcal{D}\frac{\delta H}{\delta u}$$

and to this aim we are computing consistent approximations of $\mathcal{D}$ and of the variational derivative

$$\frac{\delta H}{\delta u}.$$

The variational derivative by definition satisfies

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathcal{H}[u+\epsilon v] = \langle \frac{\delta H}{\delta u}, v\rangle_{L_2}.$$

Suppose we discretize the Hamiltonian by $\Delta x \bar{H} \approx \mathcal{H}$, and suppose we denote with $\Delta x\langle\cdot,\cdot\rangle_{\mathbf{R}^K}$ the corresponding approximation of the L_2 inner product:

$$\Delta x\langle\cdot,\cdot\rangle_{\mathbf{R}^K} \approx \langle\cdot,\cdot\rangle_{L_2}.$$

Here $\langle\cdot,\cdot\rangle_{\mathbf{R}^K}$ is the usual Euclidean inner product in $\mathbf{R}^K$. Then to obtain a consistent approximation of the variational derivative we take variations of the discrete Hamiltonian function. For example, in the case of the discretization of the Hamiltonian of the KdV equation we have

$$\Delta x \bar{H} := \Delta x \sum_{j=1}^{K}\left[\left(\frac{u_{j+1}-u_j}{\Delta x}\right)^2 - u_j^3\right]$$

and so

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \Delta x \bar{H}(u+\epsilon v) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \Delta x \sum_{j=1}^{K}\left[\left(\frac{u_{j+1}+\epsilon v_{j+1}-(u_j+\epsilon v_j)}{\Delta x}\right)^2 - (u_j+\epsilon v_j)^3\right] = ... = \Delta x\langle\nabla\bar{H}, v\rangle_{\mathbf{R}^K}$$

and the approximation of $\frac{\delta H}{\delta u}$ is then $\nabla\bar{H}$.