Structure-preserving discretization of differential equations, AARMS summer school 2015

August 5, 2015

THIRD ASSIGNMENT. This assignment is about Lie groups and Lie group integrators,

Problem 1. The bracket on a Lie algebra corresponding to a given Lie group is usually defined by imposing the Lie-Jacobi bracket on the left invariant vector fields on the Lie group. Prove that if the Lie group is $GL(d,\mathbb{R})$, the set of all real invertible $d \times d$-matrices with Lie algebra $\mathfrak{gl}(d,\mathbb{R})$, the set of all real $d \times d$ matrices, then the Lie bracket becomes the matrix commutator, i.e.

$$[\xi,\eta] = \xi \cdot \eta - \eta \cdot \xi, \quad \xi,\eta \in \ \mathfrak{gl}(d,\mathbb{R})$$

We provide some hints for your benefit

1. Prove first that all left invariant vector fields X_ξ on $GL(d,\mathbb{R})$ are of the form
$$X_\xi(A) = A\xi, \qquad A \in GL(d,\mathbb{R}),\ \xi \in \mathfrak{gl}(d,\mathbb{R})$$
2. The Lie-Jacobi bracket of two vector fields X and Y on a manifold can be locally expressed as
$$[X,Y](x) = \mathbf{D}Y(x)\cdot X(x) - \mathbf{D}X(x)\cdot Y(x)$$
Here, what is meant by the $\mathbf{D}$-operator is e.g.[1]
$$\mathbf{D}Y(x)\cdot X(x) = \left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0} Y(x+\varepsilon X(x))$$
3. Proceed to calculate
$$[\xi,\eta] = [X_\xi, X_\eta](I) = \mathbf{D}X_\eta(I)\cdot X_\xi(I) - \mathbf{D}X_\xi(I)\cdot X_\eta(I)$$

Answer:

1. We obtain the left invariant vector field as $X_\xi|_A = T_I L_A(\xi)$, so apply $\gamma(\varepsilon)$ where $\gamma(0) = I$, $\dot\gamma(0) = \xi$
$$X_\xi|_A = \left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0} A\cdot\gamma(\varepsilon) = A\xi$$

[1] If you are unfamiliar with this notation and what it means for the matrix manifold $GL(d,\mathbb{R})$, think of $X(x)$, $Y(x)$, and x as vectors of dimension d^2, then e.g. $\mathbf{D}Y(x)\cdot X(x)$ is just the Jacobian matrix ($d^2 \times d^2$) of $Y(x)$ multiplied by $X(x)$

2. This second hint is just a rewriting of the coordinate expression for the Lie-Jacobi bracket since

$$[X, Y] = \sum_{i,j} \left(X^i(x) \frac{\partial Y^j}{\partial x_i}(x) - Y^i(x) \frac{\partial X^j}{\partial x_i}(x) \right) = \mathbf{D}Y(x) \cdot X - \mathbf{D}X(x) \cdot Y.$$

3. Completing the calculation of the hint, substituting the definition of the left invariant vector field, we get

$$\begin{aligned}[\xi, \eta] &= [X_\xi, X_\eta](I) = \mathbf{D}X_\eta(I) \cdot X_\xi(I) - \mathbf{D}X_\xi(I) \cdot X_\eta(I) \\ &= \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \big((I + \varepsilon I\xi)\eta - (I + \varepsilon I\eta)\xi\big) = \xi\eta - \eta\xi\end{aligned}$$

Problem 2.

a) We consider first the Lie group action of the *affine group* G_A parametrized by pairs (A, b), where $A \in GL(d, \mathbb{R})$, $b \in \mathbb{R}^d$ on vectors x through the formula

$$(A, b) \cdot x = Ax + b \tag{1}$$

☞ Show that the group product of G_A is defined as

$$(A_1, b_1) \cdot (A_2, b_2) = (A_1 \cdot A_2, A_1 b_2 + b_1)$$

☞ What is the identity element of G_A? Find a formula for $(A, b)^{-1}$.

☞ Argue why the Lie algebra, $\mathfrak{g}_A$, of the affine group consists of pairs (ξ, v), where $\xi \in \mathfrak{gl}(d, \mathbb{R})$ and $v \in \mathbb{R}^d$, and find an expression for the commutator

$$[(\xi, v), (\eta, w)]_{\mathfrak{g}_A}$$

☞ Use the matrix exponential to express the exponential map $\exp : \mathfrak{g}_A \to G_A$.
Hint. Find the flow at $t = 1$ of the right invariant vector field on G_A applied to the identity element.

Answer:

☞ The group product is obtained by considering the axioms for the Lie group action $\Lambda(g, \Lambda(h, x)) = \Lambda(gh, x)$, using this we get

$$(A_1, b_1) \cdot (A_2, b_2) \cdot x = A_1(A_2 x + b_2) + b_1 = (A_1 A_2, A_1 b_2 + b_1) \cdot x$$

☞ The identity element (E, e) is the one that makes $(E, e) \cdot (A, b) = (A, b) \cdot (E, e) = (A, b)$ for all $(A, b) \in G_A$. From the group product we therefore find $(E, e) = (I, 0)$.
The inverse of (A, a) is the element (B, b) such that $(A, a) \cdot (B, b) = (B, b) \cdot (A, a) = (I, 0)$ which leads to

$$(B, b) = (A, a)^{-1} = (A^{-1}, -A^{-1}a)$$

☞ The Lie algebra $\mathfrak{g}_A$ consist of tangent vectors at the identity $(I, 0)$, these are tangents to all possible curves through the identity which are contained in G_A. Now, the curve $(I + \varepsilon\xi, \varepsilon v)$ belongs to G_A for *any* matrix ξ and vector v when $|\varepsilon|$ is sufficiently small since $I + \varepsilon\xi$ then is invertible. So we conclude that $\mathfrak{g}_A$ consists of all pairs (ξ, v), $\xi \in \mathfrak{gl}(d, \mathbb{R})$ and $v \in \mathbb{R}^d$. To find the bracket, one could follow exactly the same procedure with left invariant vector fields and the Lie-Jacobi bracket described in the first problem. However, a more convenient way is to consider to curves $(A(t), a(t)$ and $(B(s), b(s))$ through the identity of the group with tangents (ξ, v) and (η, w) and use the formula

$$
\begin{aligned}
[(\xi, v), (\eta, w)] &= \frac{\partial^2}{\partial t \partial s}\bigg|_{t=s=0} (A(t), a(t)) \cdot (B(s), b(s)) \cdot (A(t), a(t))^{-1} \\
&= \frac{\partial^2}{\partial t \partial s}\bigg|_{t=s=0} (A(t)B(s)A(t), -A(t)B(s)A(t)^{-1}a(t) + A(t)b(s) + a(t)) \\
&= (\dot{A}(0)\dot{B}(0) - \dot{B}(0)\dot{A}(0), -\dot{B}(0)\dot{a}(0) + \dot{A}(0)\dot{b}(0)) = ([\xi, \eta], \xi w - \eta v)
\end{aligned}
$$

☞ The right invariant vector field is obtained using the curve $(B(t), b(t))$ through $(I, 0)$ and with tangent (ξ, v) at $t = 0$.

$$X_{(\xi,v)}|_{(A,a)} = T_{(I,0)} R_{(A,a)}(\xi, v) = \frac{\partial}{\partial t}\bigg|_{t=0} (B(t), b(t)) \cdot (A, a) = (\xi A, \xi a + v)$$

So we solve the two equations $\dot{A} = \xi A$ and $\dot{a} = \xi a + v$ to obtain

$$(A(t), a(t)) = (\exp(t\xi)A_0, \exp(t\xi)a_0 + \xi^{-1}(\exp(t\xi) - I)v$$

where we note that invertibility of ξ is not required because the singularity is removable. For $(A_0, a_0) = (I, 0)$ and at $t = 1$ we get

$$\exp(\xi, v) = (\exp(\xi), \xi^{-1}(\exp(\xi) - I)v)$$

b) You shall now consider some properties of the action itself. Prove that

☞ The group action is transitive on $\mathbb{R}^d$

☞ Recall the definition of the isotropy subgroups of a Lie group G acting transitively on a manifold M

$$G_x = \{g \in G : g \cdot x = x\}$$

Prove that for any two points x, y in M the isotropy subgroups G_x and G_y are isomorphic.
Hint: Let $k \in$G be such that $y = k \cdot x$ and show that if $g \in G_y$ then $k^{-1}hk \in G_x$ etc.

☞ Find exactly what is $G_{A,x}$, $x \in \mathbb{R}^d$, and show that the isotropy groups are isomorphic to $GL(d, \mathbb{R})$.

Answer:

☞ That the action is transitive means that for any given pair $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ there is a group element $(A, a) \in G_A$ such that $y = (A, a) \cdot x = Ax + a$. This can be achieved e.g. by choosing A to be any invertible matrix and then set $a = y - Ax$.

☞ There was a misprint in the hint here, should be

...show that if $g \in G_y$ then $k^{-1}gk \in G_x$ etc.

If $g \in G_y$ it means that $g \cdot y = y$ and since $y = k \cdot x$, we have $gk \cdot x = kx$ which means that $k^{-1}gk \cdot x = x$ and $k^{-1}gk \in G_x$. Similarly, if $h \in G_x$ we find that $khk^{-1} \in G_y$, and therefore G_x and G_y are isomorphic for any x, y.

☞ With the action $(A, a) \cdot x = Ax + a$ we see that $(A, a) \in G_{A,x}$ if and only if it is of the form

$$(A, a) = (A, (I - A)x) \quad \text{for some } A \in GL(d, \mathbb{R})$$

Since all isotropy groups are isomorphic we may as well choose one of them, namely

$$G_0 = \{(A, 0) : A \in GL(d, \mathbb{R})\}$$

so they are isomorphic to $GL(d, \mathbb{R})$.

c) Consider the problem

$$\dot{q} = \frac{1}{\varepsilon}p + 4p^3 - \varepsilon q$$
$$\dot{p} = -\frac{1}{\varepsilon}q - 4q^3 - \varepsilon p$$

☞ Write the problem in the form

$$\dot{y} = F(y) = Ay + g(y), \qquad A = \begin{pmatrix} 0 & \frac{1}{\varepsilon} \\ -\frac{1}{\varepsilon} & 0 \end{pmatrix}$$

☞ Express the vector field in the form

$$F(y) = f_1(y)E_1 + f_2(y)E_2 + f_3(y)E_3$$

where

$$E_1 = p\frac{\partial}{\partial q} - q\frac{\partial}{\partial p}, \quad E_2 = \frac{\partial}{\partial q}, \quad E_3 = \frac{\partial}{\partial p}$$

☞ Implement the following commutator-free Lie group integrator for this problem

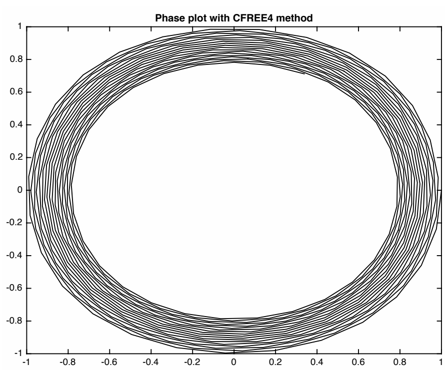
$$\begin{aligned}
Y_1 &= y_0,\\
Y_2 &= \exp(\tfrac{h}{2}k_1)\, y_0,\\
Y_3 &= \exp(\tfrac{h}{2}k_2)\, y_0\\
Y_4 &= \exp(h(k_3 - \tfrac{1}{2}k_1))\, Y_2,\\
y_{\frac{1}{2}} &= \exp(\tfrac{h}{12}(3k_1 + 2k_2 + 2k_3 - k_4))\, y_0,\\
y_1 &= \exp(\tfrac{h}{12}(-k_1 + 2k_2 + 2k_3 + 3k_4))\, y_{\frac{1}{2}}.
\end{aligned}$$

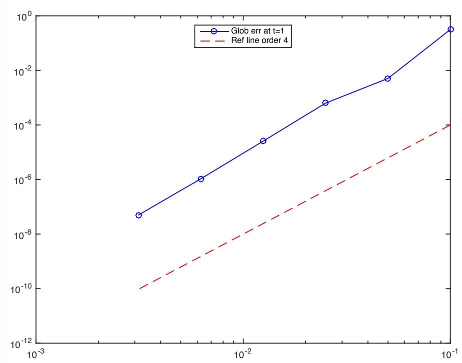
Here k_i is the ODE-vector field frozen at the point Y_i, i.e.

$$k_i = f_1(Y_i)E_1(y) + f_2(Y_i)E_2(y) + f_3(Y_i)E_3(y)$$

Use always initial value $y_0 = (1, 0)^T$.

Answer: We here show the plots from our Matlab–implementation (see separate files `commfree4.m` and `cfree4stp.m`.





☞ **To hand in from the numerical experiment.**

1. Put $\varepsilon = 0.05$, $h = 0.01$ and $T_{\text{end}} = 5$. Make a phase plot of your numerical solution to hand in.
2. Compare the numerical solution at $t = 1$ to Matlab's `ode45` using `RelTol=1e-10` and `AbsTol=1e-11`. Run the commutator free method for $h = 0.1 \cdot 2^{-k}, k = 0, \dots, 5$ and verify that your method has the correct convergence order, using e.g. a stepsize vs error `loglog` plot or a table that you hand in.

Send your answers to: elenac@math.ntnu.no