Structure-preserving discretization of differential equations, AARMS summer school 2015

July 24, 2015

THIRD ASSIGNMENT. This assignment is about Lie groups and Lie group integrators,

Problem 1. The bracket on a Lie algebra corresponding to a given Lie group is usually defined by imposing the Lie-Jacobi bracket on the left invariant vector fields on the Lie group. Prove that if the Lie group is $GL(d, \mathbb{R})$, the set of all real invertible $d \times d$-matrices with Lie algebra $\mathfrak{gl}(d, \mathbb{R})$, the set of all real $d \times d$ matrices, then the Lie bracket becomes the matrix commutator, i.e.

$$[\xi, \eta] = \xi \cdot \eta - \eta \cdot \xi, \quad \xi, \eta \in \mathfrak{gl}(d, \mathbb{R})$$

We provide some hints for your benefit

1. Prove first that all left invariant vector fields X_ξ on $GL(d, \mathbb{R})$ are of the form
$$X_\xi(A) = A\xi, \qquad A \in GL(d, \mathbb{R}),\ \xi \in \mathfrak{gl}(d, \mathbb{R})$$
2. The Lie-Jacobi bracket of two vector fields X and Y on a manifold can be locally expressed as
$$[X, Y](x) = \mathbf{D}Y(x) \cdot X(x) - \mathbf{D}X(x) \cdot Y(x)$$
Here, what is meant by the $\mathbf{D}$-operator is e.g.[1]
$$\mathbf{D}Y(x) \cdot X(x) = \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} Y(x + \varepsilon X(x))$$
3. Proceed to calculate
$$[\xi, \eta] = [X_\xi, X_\eta](I) = \mathbf{D}X_\eta(I) \cdot X_\xi(I) - \mathbf{D}X_\xi(I) \cdot X_\eta(I)$$

Problem 2.

a) We consider first the Lie group action of the *affine group* G_A parametrized by pairs (A, b), where $A \in GL(d, \mathbb{R})$, $b \in \mathbb{R}^d$ on vectors x through the formula
$$(A, b) \cdot x = Ax + b \tag{1}$$

[1] If you are unfamiliar with this notation and what it means for the matrix manifold $GL(d, \mathbb{R})$, think of $X(x)$, $Y(x)$, and x as vectors of dimension d^2, then e.g. $\mathbf{D}Y(x) \cdot X(x)$ is just the Jacobian matrix ($d^2 \times d^2$) of $Y(x)$ multiplied by $X(x)$

☞ Show that the group product of G_A is defined as

$$(A_1, b_1) \cdot (A_2, b_2) = (A_1 \cdot A_2, A_1 b_2 + b_1)$$

☞ What is the identity element of G_A? Find a formula for $(A, b)^{-1}$.

☞ Argue why the Lie algebra, $\mathfrak{g}_A$, of the affine group consists of pairs (ξ, v), where $\xi \in \mathfrak{gl}(d, \mathbb{R})$ and $v \in \mathbb{R}^d$, and find an expression for the commutator

$$[(\xi, v), (\eta, w)]_{\mathfrak{g}_A}$$

☞ Use the matrix exponential to express the exponential map $\exp : \mathfrak{g}_A \to G_A$.
Hint. Find the flow at $t = 1$ of the right invariant vector field on G_A applied to the identity element.

b) You shall now consider some properties of the action itself. Prove that

☞ The group action is transitive on $\mathbb{R}^d$

☞ Recall the definition of the isotropy subgroups of a Lie group G acting transitively on a manifold M

$$G_x = \{g \in G : g \cdot x = x\}$$

Prove that for any two points x, y in M the isotropy subgroups G_x and G_y are isomorphic.
Hint: Let $k \in$G be such that $y = k \cdot x$ and show that if $g \in G_y$ then $k^{-1}hk \in G_x$ etc.

☞ Find exactly what is $G_{A,x}$, $x \in \mathbb{R}^d$, and show that the isotropy groups are isomorphic to $GL(d, \mathbb{R})$.

c) Consider the problem

$$\dot{q} = \frac{1}{\varepsilon} p + 4p^3 - \varepsilon q$$
$$\dot{p} = -\frac{1}{\varepsilon} q - 4q^3 - \varepsilon p$$

☞ Write the problem in the form

$$\dot{y} = F(y) = Ay + g(y), \qquad A = \begin{pmatrix} 0 & \frac{1}{\varepsilon} \\ -\frac{1}{\varepsilon} & 0 \end{pmatrix}$$

☞ Express the vector field in the form

$$F(y) = f_1(y)E_1 + f_2(y)E_2 + f_3(y)E_3$$

where

$$E_1 = p\frac{\partial}{\partial q} - q\frac{\partial}{\partial p}, \quad E_2 = \frac{\partial}{\partial q}, \quad E_3 = \frac{\partial}{\partial p}$$

☞ Implement the following commutator-free Lie group integrator for this problem

$$
\begin{aligned}
Y_1 &= y_0, \\
Y_2 &= \exp(\tfrac{h}{2} k_1)\, y_0, \\
Y_3 &= \exp(\tfrac{h}{2} k_2)\, y_0 \\
Y_4 &= \exp(h(k_3 - \tfrac{1}{2} k_1))\, Y_2, \\
y_{\frac{1}{2}} &= \exp(\tfrac{h}{12}(3k_1 + 2k_2 + 2k_3 - k_4))\, y_0, \\
y_1 &= \exp(\tfrac{h}{12}(-k_1 + 2k_2 + 2k_3 + 3k_4))\, y_{\frac{1}{2}}.
\end{aligned}
$$

Here k_i is the ODE-vector field frozen at the point Y_i, i.e.

$$k_i = f_1(Y_i) E_1(y) + f_2(Y_i) E_2(y) + f_3(Y_i) E_3(y)$$

Use always initial value $y_0 = (1, 0)^T$.

☞ **To hand in from the numerical experiment.**

1. Put $\varepsilon = 0.05$, $h = 0.01$ and $T_{\text{end}} = 5$. Make a phase plot of your numerical solution to hand in.
2. Compare the numerical solution at $t = 1$ to Matlab's `ode45` using `RelTol=1e-10` and `AbsTol=1e-11`. Run the commutator free method for $h = 0.1 \cdot 2^{-k}, k = 0, \ldots, 5$ and verify that your method has the correct convergence order, using e.g. a stepsize vs error `loglog` plot or a table that you hand in.

Send your answers to: elenac@math.ntnu.no