Structure-preserving discretization of differential equations, AARMS summer school 2015

July 16, 2015

SECOND ASSIGNMENT. You have learned about discrete gradient methods for preserving invariants in ODEs. In this assignment you shall apply the same technique for a PDE. But it is then necessary to give some background information first. For a PDE, the first integral is usually an integral over the whole spatial domain of a function that depends not only on the solution $u(x,t)$, but also on its spatial derivatives, u_x, u_{xx} etc, so for instance with one space dimension we could have something like

$$H[u] = \int_{\mathbb{R}} G(u(x,t), u_x(x,t), u_{xx}(x,t), \ldots)\, \mathrm{d}x$$

What was in the ODE case the gradient of H is now replaced by the *variational derivative* $\frac{\delta H}{\delta u}$ which is defined through the relation

$$\left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} H[u+\varepsilon v] = \left\langle \frac{\delta H}{\delta u}, v \right\rangle_{L^2}$$

Looking at the example

$$H[u] = \int_{\mathbb{R}} \left(\frac{1}{2}u_x^2 - u^3\right) \mathrm{d}x \tag{1}$$

you may verify, using integration by parts and the fact that u must vanish at infinity

$$\frac{\delta H}{\delta u} = -u_{xx} - 3u^2 \tag{2}$$

The skew-symmetric matrix you saw in the ODE case has now been replaced by an operator which is skew-symmetric with respect to the L^2 inner product. For instance, one could choose

$$\mathcal{S} = \frac{\partial}{\partial x} \tag{3}$$

which is skew-symmetric with respect to the L^2 inner product. So we have arrived at the PDE itself, which is now written

$$u_t = \mathcal{S}\,\frac{\delta H}{\delta u} \tag{4}$$

Now, using the examples suggested above with invariant (1), and skew-symmetric operator (3), we would get, using (2) and (4), the KdV equation

$$u_t = -6uu_x - u_{xxx} \tag{5}$$

We shall be using the KdV equation as our case in this assignment. To make it simpler to implement, we also assume that we have periodic boundary conditions, i.e.

$$u(-L,t) = u(L,t), \quad \text{for all } t$$

and an initial condition of the form

$$u(x,0) = \frac{c}{2}\text{sech}^2\left(\frac{\sqrt{c}}{2}x\right)$$

Then the exact solution is $u(x,t) = \frac{c}{2}\text{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)$

Exercise.

1. Introduce a (uniform) grid in space $x_k = -L + k\Delta x$, $1 \leq k \leq K$, where $h = \frac{2L}{K}$.
2. Discretize the first integral H from (1) on this grid by
   (a) using a quadrature rule for the integral (rectangular rule is sufficiently accurate)
   (b) replacing the partial derivative in the integrand by finite difference quotients

   The result is a function $H_d(\mathbf{u})$, where $\mathbf{u} \in \mathbb{R}^K$
3. Compute $\nabla H_d(\mathbf{u})$
4. Discretize the operator $\mathcal{S}$ on the grid in such a way that a skew-symmetric matrix S is obtained
5. You now solve the system of ODEs

   $$\mathbf{u}_t = S\nabla H_d(\mathbf{u})$$

   by an integral preserving method such as the AVF method
6. Implement this in Matlab or any other choice of programming language.

Suggested parameters: $K = 100$, $L = 10$, $c = 4$, $h = \Delta t = 0.05$.

To be handed in.

1. Document what you have done in 1–5 above, where you briefly describe how you have discretized H, your resulting H_d, write up $\nabla H_d(\mathbf{u})$. Give your skew-symmetric matrix S_d. Which is your choice of discrete gradient, give your $\overline{\nabla} H_d(\mathbf{u},\mathbf{v})$, and how does the complete method look like.
2. Supply a plot with linear axes where you show the numerical vs the exact solution at times $t = 1$ and $t = 50$.
3. Also hand in a plot which shows $H_d(\mathbf{u}^n)$ as a function of time ($t_n = n\Delta t$) for $n = 1000$ steps with stepsize $\Delta t = 0.05$.

Send your answers by e-mail to elenac@math.ntnu.no by Wednesday July 22nd, 15:00 pm.