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Examination paper for **TMA4195 Mathematical Modeling**

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Problem 1 The length L of the ballistic trajectory of a bullet is assumed to satisfy

$$L = F(u_0, A, m, \alpha, g, \rho),$$

where u_0 and α are the exit speed and angle, A the cross-sectional area, and m the mass of the bullet, g the gravitational acceleration, and ρ the density of air.

What can you say about the function F based on dimensional analysis?

Problem 2 Consider the initial value problem

$$\epsilon y'' + y' + y^2 = 0, \quad 0 < x < 1, \quad 0 < \epsilon \ll 1; \qquad y(0) = 0, \qquad y(1) = -\frac{1}{2}.$$

Find the leading order singular perturbation approximation of $y(x)$ valid for all $0 \le x \le 1$.

Hint: The boundary layer is near $x = 0$.

Problem 3 Consider the differential equation

$$y' = (y^2 - \mu + 1)(y^2 - \mu y), \quad t > 0, \quad \mu \in \mathbb{R}. \tag{1}$$

Sketch the bifurcation diagram of (1) using solid curves to denote stable and dashed curves to denote unstable equilibrium points.

Find the bifurcation points.

Problem 4 Find the solution $\rho(x, t)$ of the conservation law

$$\rho_t + \left(\frac{1}{3}\rho^3\right)_x = 0, \quad x \in \mathbb{R}, \quad t > 0,$$

and the initial condition

$$\rho(x, 0) = \begin{cases} 2, & x < 0, \\ 1, & x > 0. \end{cases}$$

Hint: Start with the method of characteristics.

Problem 5 The current Ebola outbreak in West Africa can be modelled analogously to a chain of elementary chemical reactions:

$$S + I \xrightarrow{a} 2I, \qquad I \xrightarrow{b} R, \qquad I \xrightarrow{d} D,$$

where S, I, R, and D denote susceptible (can be infected), infected, resistent (immune to infection) and dead people, and $a, b, d > 0$ are rate constants.

Explain briefly how these reaction relations should be interpreted here.

Use the law of mass action and mass balance to derive differential equations for $S(t)$, $I(t)$, $R(t)$, and $D(t)$, the sizes of the susceptible, infected, resistent and dead parts of the population.

Problem 6 We consider a 2D model of porous soil in the domain $0 \leq x^* \leq L$, $0 \leq z^* \leq H$, between impermeable bedrock and the surface. A ground water reservoir is situated in the domain

$$\Omega^*(t^*) = \{(x, z) : 0 \leq x \leq L,\ 0 \leq z \leq h^*(x, t^*)\},$$

where the curve $z^* = h^*(x^*, t^*) \leq H$ is the water table. Water flows only in the pore volume, a constant fraction $0 < \phi < 1$ of the total volume. Let ρ be the constant mass density of water and $\vec{j}^*(x^*, z^*, t^*)$ the flux of water in the soil (the volume flow rate of water per unit area of soil). By Darcy's law,

$$\vec{j^*} = -\frac{K}{\mu}\nabla(p^* + \rho g z^*),$$

where $p^*(x^*, z^*, t^*)$ is the pressure and the constants $K, \mu, g > 0$ are the permeability, water viscosity, and gravitational accelleration.

a) Let R be any nice and fixed domain (a control volume) in $\Omega^*(t^*)$. Explain why

$$\frac{d}{dt^*}\int_R \phi\rho\, dx^* dz^* = \int_{\partial R} \frac{K}{\mu}\nabla(p^* + \rho g z^*) \cdot \vec{n}\, d\sigma.$$

Use this result to derive the following conservation law in differential form,

$$\frac{\partial^2 p^*}{\partial x^{*2}} + \frac{\partial^2 p^*}{\partial z^{*2}} = 0 \qquad \text{in} \qquad \Omega^*(t^*).$$

Hint: You may assume that p^* is smooth.

To solve the equation, we need the initial height h_0, boundary conditions for p^* along the boundary of $\Omega^*(t^*)$, and an additional condition at the water table:

$$\frac{\mu\phi}{K} h^*_{t^*} - h^*_{x^*} p^*_{x^*} = -p^*_{z^*} - \rho g \qquad \text{at} \qquad z^* = h^*(x^*, t^*). \tag{2}$$

b) Find natural scales for x^*, z^*, and h^*.

Use equation (2) to find a scale for t^* when the scale for p^* is $P = \rho g H$ and $H \ll L$.

We go back to unscaled quantities and aim to derive a new model under the simplifying assumption that the pressure is hydrostatic:

$$p^*(x^*, z^*, t^*) = \rho g \left(h^*(x^*, t^*) - z^* \right) \qquad \text{for} \qquad 0 \le z^* \le h^*(x^*, t^*).$$

c) Show that the height $h^*(x^*, t^*)$ of the water table then satisfies

$$\frac{\partial h^*}{\partial t^*} = C \frac{\partial}{\partial x^*} \left(h^* \frac{\partial h^*}{\partial x^*} \right) \qquad \text{in} \qquad x^* \in (0, L), \quad t^* > 0,$$

for some constant C.

Hint: You may reduce the problem to one space dimension by integrating densities and fluxes with respect to z^*. You may also assume that h^* is smooth.