



**NTNU – Trondheim**  
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Department of Mathematical Sciences

## Examination paper for **TMA4195 Mathematical Modeling**

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**Examination date:** December 11, 2014

**Examination time (from–to):** 9:00–13:00

**Permitted examination support material:**

C: Approved simple calculator, Rottman: *Matematisk formelsamling*.

**Language:** English

**Number of pages:** 3

**Number pages enclosed:** 0

**Checked by:**

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Date

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**Problem 1** The length  $L$  of the ballistic trajectory of a bullet is assumed to satisfy

$$L = F(u_0, A, m, \alpha, g, \rho),$$

where  $u_0$  and  $\alpha$  are the exit speed and angle,  $A$  the cross-sectional area, and  $m$  the mass of the bullet,  $g$  the gravitational acceleration, and  $\rho$  the density of air.

What can you say about the function  $F$  based on dimensional analysis?

**Problem 2** Consider the initial value problem

$$\epsilon y'' + y' + y^2 = 0, \quad 0 < x < 1, \quad 0 < \epsilon \ll 1; \quad y(0) = 0, \quad y(1) = -\frac{1}{2}.$$

Find the leading order singular perturbation approximation of  $y(x)$  valid for all  $0 \leq x \leq 1$ .

*Hint:* The boundary layer is near  $x = 0$ .

**Problem 3** Consider the differential equation

$$(1) \quad y' = (y^2 - \mu + 1)(y^2 - \mu y), \quad t > 0, \quad \mu \in \mathbb{R}.$$

Sketch the bifurcation diagram of (1) using solid curves to denote stable and dashed curves to denote unstable equilibrium points.

Find the bifurcation points.

**Problem 4** Find the solution  $\rho(x, t)$  of the conservation law

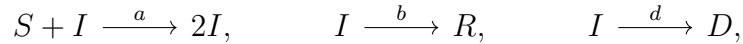
$$\rho_t + \left(\frac{1}{3}\rho^3\right)_x = 0, \quad x \in \mathbb{R}, \quad t > 0,$$

and the initial condition

$$\rho(x, 0) = \begin{cases} 2, & x < 0, \\ 1, & x > 0. \end{cases}$$

*Hint:* Start with the method of characteristics.

**Problem 5** The current Ebola outbreak in West Africa can be modelled analogously to a chain of elementary chemical reactions:



where  $S$ ,  $I$ ,  $R$ , and  $D$  denote susceptible (can be infected), infected, resistant (immune to infection) and dead people, and  $a, b, d > 0$  are rate constants.

Explain briefly how these reaction relations should be interpreted here.

Use the law of mass action and mass balance to derive differential equations for  $S(t)$ ,  $I(t)$ ,  $R(t)$ , and  $D(t)$ , the sizes of the susceptible, infected, resistant and dead parts of the population.

**Problem 6** We consider a 2D model of porous soil in the domain  $0 \leq x^* \leq L$ ,  $0 \leq z^* \leq H$ , between impermeable bedrock and the surface. A ground water reservoir is situated in the domain

$$\Omega^*(t^*) = \{(x, z) : 0 \leq x \leq L, 0 \leq z \leq h^*(x, t^*)\},$$

where the curve  $z^* = h^*(x^*, t^*) \leq H$  is the water table. Water flows only in the pore volume, a constant fraction  $0 < \phi < 1$  of the total volume. Let  $\rho$  be the constant mass density of water and  $\vec{j}^*(x^*, z^*, t^*)$  the flux of water in the soil (the volume flow rate of water per unit area of soil). By Darcy's law,

$$\vec{j}^* = -\frac{K}{\mu} \nabla(p^* + \rho g z^*),$$

where  $p^*(x^*, z^*, t^*)$  is the pressure and the constants  $K, \mu, g > 0$  are the permeability, water viscosity, and gravitational acceleration.

- a) Let  $R$  be any nice and fixed domain (a control volume) in  $\Omega^*(t^*)$ . Explain why

$$\frac{d}{dt^*} \int_R \phi \rho dx^* dz^* = \int_{\partial R} \frac{K}{\mu} \nabla(p^* + \rho g z^*) \cdot \vec{n} d\sigma.$$

Use this result to derive the following conservation law in differential form,

$$\frac{\partial^2 p^*}{\partial x^{*2}} + \frac{\partial^2 p^*}{\partial z^{*2}} = 0 \quad \text{in} \quad \Omega^*(t^*).$$

*Hint:* You may assume that  $p^*$  is smooth.

To solve the equation, we need the initial height  $h_0$ , boundary conditions for  $p^*$  along the boundary of  $\Omega^*(t^*)$ , and an additional condition at the water table:

$$(2) \quad \frac{\mu\phi}{K} h_{t^*}^* - h_{x^*}^* p_{x^*}^* = -p_{z^*}^* - \rho g \quad \text{at} \quad z^* = h^*(x^*, t^*).$$

**b)** Find natural scales for  $x^*$ ,  $z^*$ , and  $h^*$ .

Use equation (2) to find a scale for  $t^*$  when the scale for  $p^*$  is  $P = \rho g H$  and  $H \ll L$ .

We go back to unscaled quantities and aim to derive a new model under the simplifying assumption that the pressure is hydrostatic:

$$p^*(x^*, z^*, t^*) = \rho g (h^*(x^*, t^*) - z^*) \quad \text{for} \quad 0 \leq z^* \leq h^*(x^*, t^*).$$

**c)** Show that the height  $h^*(x^*, t^*)$  of the water table then satisfies

$$\frac{\partial h^*}{\partial t^*} = C \frac{\partial}{\partial x^*} \left( h^* \frac{\partial h^*}{\partial x^*} \right) \quad \text{in} \quad x^* \in (0, L), \quad t^* > 0,$$

for some constant  $C$ .

*Hint:* You may reduce the problem to one space dimension by integrating densities and fluxes with respect to  $z^*$ . You may also assume that  $h^*$  is smooth.