Department of Mathematical Sciences

Exam in TMA4195 Mathematical Modeling 11.12.2014 Solutions

Problem 1

The dimensional matrix A is

$$\begin{array}{c|ccccccc} & L & u_0 & A & m & \alpha & g & \rho \\ \hline \mathrm{m} & 1 & 1 & 2 & 0 & 0 & 1 & -3 \\ \mathrm{s} & 0 & -1 & 0 & 0 & 0 & -2 & 0 \\ \mathrm{kg} & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}.$$

It can easily be checked that A has rank 3, and so by Buckingham's Pi Theorem, there are exactly $7-3=4$ dimensionally independent combinations. Since we want L as a function of the other variables, we exclude it as a core variable; for simplicity, we choose the core variables as u_0, A and m. The dimensionless combinations are thus as follows:

$$\pi_1 = \frac{L}{\sqrt{A}}, \qquad \pi_2 = \alpha, \qquad \pi_3 = \frac{g\sqrt{A}}{u_0^2} \qquad \text{and} \qquad \pi_4 = \frac{\rho A^{\frac{3}{2}}}{m}.$$

Furthermore, Buckingham's Pi Theorem states that any physical relation

$$G(L, u_0, A, m, \alpha, g, \rho) = 0$$

is equivalent to a relation between the associated dimensionless combinations:

$$\Psi(\pi_1, \pi_2, \pi_3, \pi_4) = 0.$$

Assuming that this equation allows us to solve with respect to π_1, we find

$$\begin{aligned} \pi_1 &= \Phi(\pi_2, \pi_3, \pi_4) \\ \Rightarrow L &= \sqrt{A}\Phi\left(\alpha, \frac{g\sqrt{A}}{u_0^2}, \frac{\rho A^{\frac{3}{2}}}{m}\right). \end{aligned}$$

In other words, we know that there exists a function Φ such that

$$F(u_0, A, m, \alpha, g, \rho) = \sqrt{A}\Phi\left(\alpha, \frac{g\sqrt{A}}{u_0^2}, \frac{\rho A^{\frac{3}{2}}}{m}\right).$$

Note: Other (equivalent!) relations can be given using other choices of dimensionless combinations.

Problem 2

We start by calculating the outer solution y_O. We set $\epsilon = 0$ and obtain the equation

$$y_O' + y_O^2 = 0, \qquad y_O(1) = -\frac{1}{2},$$

where we have used the hint that the boundary layer is located near $x = 0$, meaning that the outer equation must satisfy the boundary condition at $x = 1$. We solve this by assuming $y_O \neq 0$ and dividing by y_O^2 to find

$$\begin{aligned} \frac{y_O'}{y_O^2} &= -1 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{y_O}\right) &= -1 \\ \Rightarrow \frac{1}{y_O} &= x + C \\ \Rightarrow y_O &= \frac{1}{x+C}. \end{aligned}$$

Imposing the boundary condition, we obtain the outer solution:

$$y_O = \frac{1}{x-3}.$$

To find the inner solution, we must first obtain a consistent scaling in the boundary layer. We scale the x axis by $x = \delta\xi$ and obtain the rescaled equation, with $Y = Y(\xi)$

$$\frac{\epsilon}{\delta^2}Y'' + \frac{1}{\delta}Y' + Y^2 = 0$$

Assuming $Y, Y', Y'' \sim 1$ in this area, we use the method of dominant balance to determine the scaling of δ. There are three choices: $\delta = \epsilon, \delta = \sqrt{\epsilon}$ and $\delta = 1$. Choosing $\delta = 1$ yields the original equation, which is uninteresting. Choosing $\delta = \sqrt{\epsilon}$ causes the second term to become dominant, and is an inconsistent approximation. However, choosing $\delta = \epsilon$, we get the equation

$$Y'' + Y' + \epsilon Y^2 = 0,$$

which is consistent and allows us to disregard the Y^2 term. We thereby get the inner equation

$$Y_I'' + Y_I' = 0, \qquad Y_I(0) = 0,$$

with general solution

$$Y_I(\xi) = A + B\mathrm{e}^{-\xi}$$

and, with the boundary condition imposed,

$$Y_I(\xi) = A(1 - \mathrm{e}^{-\xi})$$

Next, using the matching condition, we get

$$A = \lim_{\xi \to \infty} Y_I(\xi) = \lim_{x \to 0} y_O(x) = -\frac{1}{3}$$

Finally, combining the inner and outer solution and subtracting the matching constant, we get the uniform solution

$$\begin{aligned} y_U(x) &= y_O(x) + Y_I\left(\frac{x}{\epsilon}\right) - \lim_{\xi \to \infty} Y_I(\xi) \\ \Rightarrow y_U(x) &= \frac{1}{x-3} + \frac{1}{3}\mathrm{e}^{-\frac{x}{\epsilon}}. \end{aligned}$$

Problem 3

We find the equilibrium points as the solutions of

$$f(y) = y' = y(y - \sqrt{\mu - 1})(y + \sqrt{\mu - 1})(y - \mu) = 0,$$

i.e. the equilibrium points are $y \in \{0, \sqrt{\mu - 1}, -\sqrt{\mu - 1}, \mu\}$. Note that the equilibrium points $y = \sqrt{\mu - 1}$ and $y = -\sqrt{\mu - 1}$ are real only for $\mu \geq 1$.

To investigate their stability property with respect to μ, we look at the sign of $f'(y)$ at each equilibrium point as μ changes. A negative sign implies stability, while a positive sign implies instability. Firstly, we have that

$$\begin{aligned} f'(y) = (y - \sqrt{\mu - 1})(y + \sqrt{\mu - 1})(y - \mu) + y(y + \sqrt{\mu - 1})(y - \mu) \\ ... + y(y - \sqrt{\mu - 1})(y - \mu) + y(y - \sqrt{\mu - 1})(y + \sqrt{\mu - 1}). \end{aligned}$$

We now observe that

$$f'(0) = (\mu - 1)\mu \quad \begin{cases} > 0, & \mu > 1 \\ < 0, & 0 < \mu < 1 \\ > 0, & \mu < 0 \end{cases}$$

$$f'(\mu) = \mu(\mu^2 - \mu + 1) \quad \begin{cases} > 0, & \mu > 0 \\ < 0, & \mu < 0 \end{cases}$$

$$f'(\pm\sqrt{\mu - 1}) = 2\sqrt{\mu - 1}(\sqrt{\mu - 1} - \mu) < 0, \quad \mu > 1$$

From this, we get the bifurcation diagram shown in figure 1. The bifurcation points are (0,0) and (0,1), and we can see that the equilibrium solutions change stability when passing through these points.

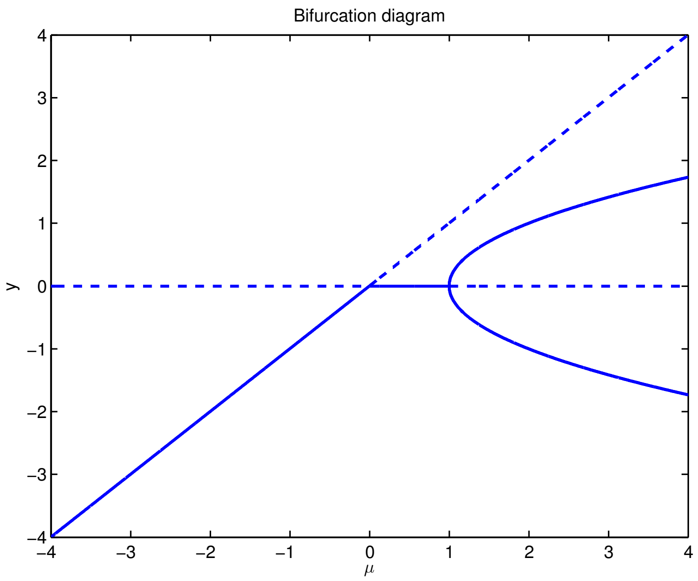

Figure 1: Bifurcation diagram.

Problem 4

We rewrite the equation in standard form:

$$\rho_t + \rho^2 \rho_x = 0, \qquad x \in \mathbb{R}, \quad t > 0$$

and solve it by the method of characteristics. Introducing $z(t) = \rho(x(t), t)$, we see that

$$\dot{z} = \rho_t + \dot{x}\rho_x,$$

and choosing $\dot{x} = z^2$ yields the system of ODES:

$$\begin{aligned} &\dot{x} = z^2, \quad x(0) = x_0 \\ &\dot{z} = 0, \qquad z(0) = \rho(x(0), 0) = \rho(x_0, 0), \end{aligned}$$

with solutions:

$$\begin{aligned} x(t) &= \rho(x_0, 0)^2 t + x_0 \\ z(t) &= \rho(x_0, 0). \end{aligned}$$

Using the initial conditions, we get two families of characteristics:

$$x(t) = \begin{cases} 4t + x_0, & x_0 < 0 \\ t + x_0, & x_0 > 0. \end{cases}$$

Since the cinematic velocity $c(\rho) = \rho^2$ is greater for characteristics starting at $x_0 < 0$ than for those starting at $x_0 > 0$, the solution will develop a shock, starting at $x = 0$ at $t = 0$. The speed of this shock is determined by the Rankine-Hugoniot condition:

$$\begin{aligned} \dot{S}(t) &= \frac{j(\rho^+) - j(\rho^-)}{\rho^+ - \rho^-} = \frac{\frac{1}{3}(\rho^+)^3 - \frac{1}{3}(\rho^-)^3}{\rho^+ - \rho^-} = \frac{\frac{1}{3}2^3 - \frac{1}{3}1^3}{2-1} = \frac{7}{3} \\ \Rightarrow S(t) &= \frac{7}{3}t \end{aligned}$$

The characteristics and the shock are shown in figure 2.

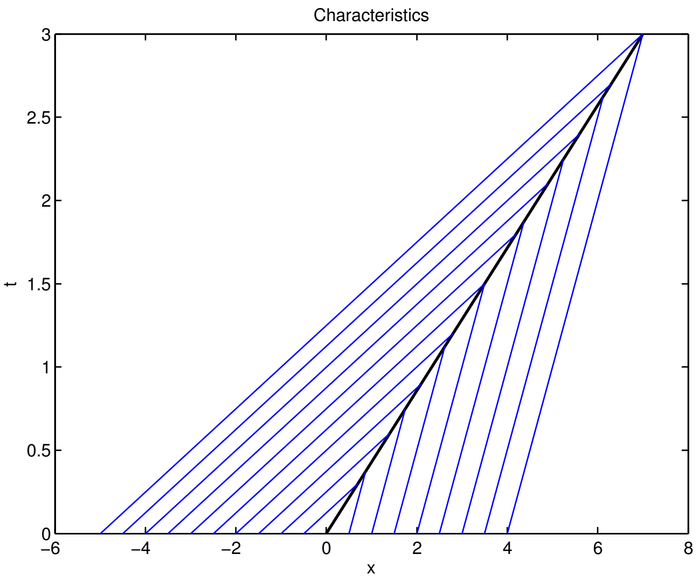

Figure 2: Characteristics and shock.

We may summarize the solution as:

$$\rho(x,t) = \begin{cases} 2, & x < \frac{7}{3}t \\ 1, & x > \frac{7}{3}t. \end{cases}$$

Problem 5

Interpretation of reactions:

- When an infected and a susceptible person interacts, there is a chance of the susceptible person getting infected.

- Infected people have a chance of recovering.
- Infected people have a chance of dying.

Consumption, production and reaction rates:

- In the first reaction, one S is consumed and one I is produced. Reaction rate: $r_a = aSI$.
- In the second reaction, one I is consumed and one R is produced. Reaction rate: $r_b = bI$.
- In the third reaction, one I is consumed and one D is produced. Reaction rate: $r_c = cI$.

Disregarding births and deaths due to other circumstances, the total amount of people ($S + I + R + D$) must be constant. We can then set up the system of ODEs governing the evolution of the populations:

$$\begin{aligned}
\dot{S} &= -r_a = -aSI \\
\dot{I} &= r_a - r_b - r_c = aSI - bI - dI \\
\dot{R} &= r_b = bI \\
\dot{D} &= r_c = dI
\end{aligned}$$

Problem 6

a) The total mass of water in R is equal to the density times the available volume:

$$\begin{aligned}
\text{total mass} &= \rho\phi \int\limits_R \mathrm{d}x^* \mathrm{d}z^* \\
&= \int\limits_R \rho\phi \mathrm{d}x^* \mathrm{d}z^*.
\end{aligned}$$

The general conservation law states that

$$\frac{\text{change of mass in R}}{\text{time}} = -\text{flux out of R} + \text{production in R},$$

or:

$$\frac{\mathrm{d}}{\mathrm{d}t^*} \int\limits_R \rho\phi \mathrm{d}x^* \mathrm{d}z^* = -\int\limits_{\partial R} \mathbf{j} \cdot \mathbf{n} \mathrm{d}\sigma + \int\limits_R q(x^*, t^*) \mathrm{d}x^* \mathrm{d}z^*.$$

We may now note that there is no production in the domain, i.e. $q(x^*, t^*) \equiv 0$, and use Darcy's law to express $\mathbf{j}$, yielding

$$\frac{\mathrm{d}}{\mathrm{d}t^*} \int\limits_R \rho\phi \mathrm{d}x^* \mathrm{d}z^* = \int\limits_{\partial R} \frac{K}{\mu} \nabla(p^* + \rho g z^*) \cdot \mathbf{n} \mathrm{d}\sigma.$$

Since ϕ, ρ and R are constant, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t^*} \int\limits_R \rho\phi \mathrm{d}x^* \mathrm{d}z^* = 0.$$

Furthermore, we can apply the divergence theorem,

$$\int\limits_{\partial R} \mathbf{j} \cdot \mathbf{n} \mathrm{d}\sigma = \int\limits_R \nabla \cdot \mathbf{j} \mathrm{d}x^* \mathrm{d}z^*,$$

to obtain the equation

$$0 = \frac{K}{\mu} \int\limits_R \nabla \cdot \nabla(p^* + \rho g z^*) \mathrm{d}x^* \mathrm{d}z^* = \frac{K}{\mu} \int\limits_R \nabla^2 p^* \mathrm{d}x^* \mathrm{d}z^*. \tag{1}$$

We now fix an arbitrary point $(x_0, z_0) \in \Omega^*(t^*)$ and choose

$$R = R_r = \{(x^*, z^*) : |x^* - x_0| < \frac{r}{2}, |z^* - z_0| < \frac{r}{2}\},$$

and note that in R_r, since $\nabla^2 p^*$ is continuous, we have

$$\nabla^2 p^*(x^*, z^*) = \nabla^2 p^*(x_0, z_0) + o(1) \text{ as } r \to 0.$$

Hence, inserting this into equation (1), we get

$$0 = \frac{1}{\int\limits_R \mathrm{d}x^* \mathrm{d}z^*} [\nabla^2 p^*(x_0, z_0) + o(1)] \int\limits_R \mathrm{d}x^* \mathrm{d}z^* = \nabla^2 p^*(x_0, z_0) + o(1),$$

and finally, letting $r \to 0$ and emphasizing that (x_0, z_0) was chosen arbitrarily, we get

$$\nabla^2 p^* = \frac{\partial^2 p^*}{\partial x^{*2}} + \frac{\partial^2 p^*}{\partial z^{*2}} = 0 \qquad \text{in} \qquad \Omega^*(t^*)$$

Note: Another way of arriving at the same conclusion is to observe that since the control volume R was chosen arbitrarily, and since p^* is assumed smooth, equation (1) can only hold if the integrand is zero everywhere in $\Omega^*(t^*)$

b) Natural scalings for x^*, z^* and h^* are L, H and H, respectively. Inserting the scaled variables $x^* = Lx, z^* = Hz, h^* = Hh, p^* = \rho g H p$, and $t^* = Tt$ into the equation, we get

$$\frac{\mu\phi H}{KT} h_t - \frac{\rho g H^2}{L^2} h_x p_x = -\rho g p_z - \rho g$$
$$\Rightarrow \frac{\mu\phi H}{\rho g K T} h_t - \frac{H^2}{L^2} h_x p_x = -(p_z + 1).$$

Since $h, p \in (0,1)$, the right hand side is ~ 1, and the second term on the left hand side is negligible. We therefore wish to choose T such that

$$\frac{\mu\phi H}{\rho g K T} = 1 \quad \Rightarrow \quad T = \frac{\mu\phi H}{\rho g K}.$$

c) We follow the hint and reduce the problem to one space dimension by introducing the new variables:

$$\varphi(x^*, t^*) = \rho\phi h^*(x^*, t^*) = \frac{\text{mass of water at (x,t)}}{\text{length}}$$
$$Q(x^*, t^*) = \int_0^{h^*(x^*,t^*)} \mathbf{j}(x^*, t^*, z) \cdot \mathbf{e}_x \mathrm{d}z = \frac{\text{volume flow rate through } x^* \text{ at time } t^*}{\text{time}}.$$

Using Darcy's law and the assumption of hydrostatic pressure, we have

$$\mathbf{j}(x^*, t^*, z) \cdot \mathbf{e}_x = -\frac{K}{\mu} p^*_{x^*}(x^*, t^*, z) = -\frac{K}{\mu} h^*_{x^*}(x^*, t^*)$$
$$\Rightarrow \quad Q(x^*, t^*) = -(h^* h^*_{x^*})(x^*, t^*)$$

We now let $x^* \in (0, L)$ and $t^* > 0$ and set up the conservation law for water in the interval $(x^*, x^* + \Delta x^*)$:

$$\frac{\mathrm{d}}{\mathrm{d}t^*} \int_{x^*}^{x^*+\Delta x^*} \varphi(x, t^*) \mathrm{d}x = Q(x^*, t) - Q(x^* + \Delta x^*, t). \tag{2}$$

Now, since φ is smooth, we have that:

$$\frac{\mathrm{d}}{\mathrm{d}t^*} \int_{x^*}^{x^*+\Delta x^*} \varphi(x, t^*) \mathrm{d}x = \int_{x^*}^{x^*+\Delta x^*} \varphi_{t^*}(x, t^*) \mathrm{d}x = \Delta x^* (\varphi_{t^*}(x^*, t^*) + o(1))$$

as $\Delta x^* \to 0$. Inserting this into (2), dividing by Δx^* and letting $\Delta x^* \to 0$, we get

$$\varphi_{t^*} = -Q_{x^*}(x^*, t^*)$$
$$\Rightarrow \quad h^*_{t^*} = \frac{K}{\rho g \mu} \frac{\partial}{\partial x^*}(h^* h^*_{x^*}) \quad x^* \in (0, L), \quad t^* > 0.$$