Probability, Sufficiency, and the Gamma Distribution

Gunnar.Taraldsen@ntnu.no
Department of Mathematical Sciences
Norwegian University of Science and Technology

September 1, 2022

Abstract

The gamma distribution and the concept of sufficiency are important in statistical inference. Key elements of this are demonstrated by solving the problems formulated in this note. This requires some knowledge of probability theory, statistics, and multivariate calculus. Casella and Berger (2002) give many more illustrations and explanations of the theory behind the methods of statistical inference. [1]

Contents

[1] 2020 Mathematics Subject Classification: 62-01 Introductory exposition pertaining to statistics; 62B05 Sufficient statistics and fields; 62E10 Characterization and structure theory of statistical distributions;

1 Probability

It is, as always, assumed that Ω is a fixed abstract underlying probability space with probability measure P and family of events $\mathcal{E}$ (a sigma algebra).

1.1 Random variables

A function $X : \Omega \to \mathbb{R}$ is a random variable if $(X \leq x) = \{\omega \,|\, X(\omega) \leq x\}$ is an event for all real x. Let X and Y be random variables and let t be a constant. Use this and the defining properties of events to prove that t, $X + Y$, XY, and $\exp(tX)$ are random variables. Why and how is the cumulative distribution function of a random variable always well defined? Prove that the set of random variables is a vector space.

1.2 Random quantities

Let $X : \Omega \to \mathcal{X}$ be a function. Prove that the family of sets E such that $(X \in E) = \{\omega \,|\, X(\omega) \in E\}$ is an event is a σ-algebra (sigma algebra). Prove that $\mathrm{P}_X(E) = \mathrm{P}(X \in E)$ defines a probability measure (probability function) P_X. A function $X : \Omega \to \Omega_X$ is by definition measurable if $(X \in A)$ is an event whenever A is an event. Prove that $\mathrm{P}_X(A) = \mathrm{P}(X \in A)$ defines a probability measure (probability function) P_X. What is the difference between the two given definitions of the law P_X of X?

1.3 Borel field

In the exercises you proved that the intersection of two σ-algebras is a σ-algebra. Prove that the intersection of an arbitrary collection of σ-algebras is a σ-algebra. The Borel field of the real line is the intersection of the collection of σ-algebras that contain the open intervals. A set is a Borel set if it belongs to the Borel field. Prove that $\mathrm{P}_X(B)$ is well defined if B is a Borel set and X is a random variable.

2 The Gamma Distribution

The notation $X \sim \mathbf{Gamma}(\alpha, \beta)$ means that $X > 0$ is a gamma distributed random variable with shape $\alpha > 0$ and scale $\beta > 0$. The density of $X \sim \mathbf{Gamma}(\alpha) = \mathbf{Gamma}(\alpha, 1)$ is proportional to $x^{\alpha-1}e^{-x}$.

2.1 Moment Generating Function

Find the density of $X \sim \mathbf{Gamma}(\alpha, \beta) = \beta\,\mathbf{Gamma}(\alpha)$. The moment generating function M of X is defined by $M(t) = \mathrm{E}e^{tX}$. Find the moment generating function of $X \sim \mathbf{Gamma}(\alpha)$. Find the moment generating function of $X \sim \mathbf{Gamma}(\alpha, \beta)$. How is the probability exercise and the change of variables theorem relevant here?

2.2 Sum

Assume $S = X_1 + \cdots + X_n$ where $X_i \sim \textbf{Gamma}(\alpha_i)$ are independent. Find the moment generating function and the probability density of S. Find the moment generating function and the probability density of S if $X_i \sim \textbf{Gamma}(\alpha_i, \beta)$.

3 Sufficiency

In the following it will be assumed throughout that the data x_1, x_2, x_3 is a random sample from $\textbf{Gamma}(\alpha, \beta)$. Furthermore, $s_1 = \overline{x}$, $s_2 = \tilde{x}$ (geometric mean), and $s_3 = w = s_2/s_1$.

3.1 Known Shape

Assume the shape α to be known. Prove that $T = S_1$ is a minimal complete sufficient statistic. Illustrate the level sets $[x]_t = \{x' \mid t(x) = t(x')\}$ in a drawing. Explain that the level sets give a partion of the data space. Prove that $[x]_L = [x]_t$ where L is the likelihood statistic. Explain that this gives a one-one correspondence between L and t. What does the sufficiency principle say in this case? Find an estimator for the unknown model parameter which do not violate the sufficiency principle. What is the uncertainty of the estimate? Find the conditional distribution of the data X given $T = t$. Use this to give an alternative proof of sufficiency of T.

3.2 The Bartlett Statistic W

Prove that $g(\mathrm{E}V) \leq \mathrm{E}g(V)$ holds when g is convex (Jensen's inequality). Use this to prove that $0 < w \leq 1$. Use the Basu theorem to prove that $\overline{X}$ and W are idependent.

3.3 Known Scale

Assume the scale β is known. Find a minimal complete sufficient T, and redo the exercise in subsection 3.1 for this case.

3.4 Unknown Shape and Scale

Assume both shape α and scale β to be unknown. Find a minimal complete sufficient T, and redo the exercise in subsection 3.1 for this case.

References

Casella, G. and R. L. Berger (2002). *Statistical Inference* (2nd edition ed.). Duxbury, Thomson learning.