1.1. Most of the solutions were ok

1.2. Proving that $\tau = \gamma'$

A possible solution

The moment generating function

$$M_T(w) = E\left(e^{wT}\right) = \int_{-\infty}^{\infty} h(x) e^{(\theta+w)t(x) - \gamma(\theta)} dx =$$

using "w", because "t" is already used

$$= e^{\gamma(\theta+w) - \gamma(\theta)} \int_{-\infty}^{\infty} h(x) e^{(\theta+w)t(x) - \gamma(\theta+w)} dx =$$

$$= e^{\gamma(\theta+w) - \gamma(\theta)}$$

$$\tau = E(T) = \left.\frac{d M_T(w)}{dw}\right|_{w=0} = e^{\gamma\theta - \gamma\theta} \cdot \frac{\partial \gamma}{\partial \theta}(\theta + 0) = \gamma' \quad \square$$

Those who obtained $\tau = \gamma'$ in 1.1. and refered to that, also got full score here.

1-1 correspondence between τ and θ

$$\frac{\partial \tau}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{\int_{-\infty}^{\infty} t e^{\theta t(x)} h(x) dx}{e^{\gamma}}\right) =$$

$$= \frac{e^{\gamma}\int_{-\infty}^{\infty} t^2 e^{\theta t(x)} h(x)\,dx - e^{\gamma}\cdot\gamma'\cdot\int_{-\infty}^{\infty} t e^{\theta t(x)} h(x)\,dx}{e^{2\gamma}} =$$

$$= \frac{e^{2\gamma} E(T^2) - e^{2\gamma}\left[E(T)\right]^2}{e^{2\gamma}} = Var(T) \geq 0$$

Except for the degenerated case where $Var(T)=0$ for some θ $\Rightarrow X$ is a constant with probability 1 $\Rightarrow X$ is a constant with probability 1 $\forall \theta \in \Omega_0$, we have

$Var(T) > 0 \Rightarrow \tau$ is a strictly increasing function of $\theta \Rightarrow$ 1-1 correspondence □

The further solution of 1.2 was mainly ok until completeness

If you show by definition that

$$E[g(T)] = 0 \Rightarrow P\big(g(T) = 0\big) = 1,$$

please explain why if

$E[g(T)] = \int_{-\infty}^{\infty} g(t) f(t|\theta)\,dt = 0$, the function g

can't be negative for some values t and positive for other t. (Nobody has explained this).

It is ok to apply the Theorem 6.2.25.

1.4. The only common mistake occured with $U((1-k)\beta, (1+k)\beta)$. It is not the same as the general $U(a, b)$, whether k was known or β was known.

It was not from a location family, for both cases.

However, for k known, and for any $a \in \mathbb{R}$,

$$a \cdot X \sim U((1-k)a\cdot\beta, (1+k)a\beta),$$

so it is the scale family

2.1, proof of Basu.

Note that your proofs should

Note that your proofs should
use completeness, sufficiency and ancillarity! If a proof is too general (f.ex. you show that the statement also holds if S is not ancillary, while it shouldn't hold) – there is a mistake.

Note also that citing an article where the Basu theorem was considered as obviously following from Eq.(2), was not enough.

One of possible solutions:

Let T be complete sufficient statistic
S be ancillary statistic

Let $$\phi(S) = I_{S\in A} = \begin{cases} 1, & S \in A, \\ 0 & \text{otherwise} \end{cases}$$

where A is an event in Ω_S

Let $$g(T) = \underbrace{\phi(S|T))}_{\substack{\text{constant with respect to } \theta \\ \text{by sufficiency of } T}} - \underbrace{\phi(S)}_{\substack{\text{constant w.r.t.. } \theta \\ \text{by ancillarity of } S}}$$

So $g(T)$ is only function of data, but not θ

By the Eq. (2) and the linearity of expectation,

$$E(g(T)) = E(\phi(S|t)) - E(\phi(S)) = 0 \quad \forall \theta$$

By completeness of T, $P(g(T)=0)=1$

So $P(S \in A | T) = P(S \in A)$,

and this holds for all sets A such that $S \in A$ is an event

$\Rightarrow S$ and T are independent $\square$

2.4.

Explaining why the identity

$l(t(\beta U), \beta) = l(t(x), xU^{-1})$ (*)

gives that the risk is minimized by the given statistic

(*) $\Rightarrow \rho = E\left[(\ln(t(x) \cdot (xU^{-1}))^2\right] = E\left[(\ln t(x) - \ln(xU^{-1}))^2\right]$

[We know that $E\left[(b-Y)^2\right]$ is minimized w.r.t. b when $b = E(Y)$]

So ρ is minimized when $\ln t(x) = E\left[\ln(xU^{-1})\right]$

$\square$

Assumption (A) means actually assumption on ekvivariant t (we could also search $t = aX$, $a \in \mathbb{R}$)

Without this assumption we might have found another statistics which give lower risk for some values of β, but much higher risk for another values of β.

It was **wrong** to:

- × set the statistic into the expression for risk and get risk $\rho = 0$ (risk was nonzero, because $E[(\ln(XU^{-1}))^2] \neq [E(\ln(XU^{-1}))]^2$)
- × investigate $\frac{\partial \rho}{\partial \beta}$

Calculating t for Gamma (α, β), α known

$X = \frac{\sum_{i=1}^{n} X_i}{n} = \beta U$, where $U \sim \text{Gamma}(n\alpha, \frac{1}{n})$.

Then $E(\ln(U)) = \psi(\alpha) - n$, where

Wikipedia page on gamma distribution

ψ is a digamma function

Hence $t = \underbrace{X \cdot \exp(-\psi(\alpha) + n)}$

note that there is no "β" in the expression!

t is a statistic, i.e. function only of data, not the unknown parameter!!!

Anything including β in this answer was a big mistake.