Plenumsregning 14: Oppsummering

Eksamen vår 2023

Oppgave 8

Finn matrisen A som tilfredsstiller likningen

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & -1 \end{bmatrix} A \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 3 \end{bmatrix}$$

Let $B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, and $D = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 3 \end{bmatrix}$.

So $BAC = D \iff A = B^{-1} D C^{-1}$

Next, we compute B^{-1} by considering the following:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & -2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - 2R_1 \to R_3]{R_2 + R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & -2 & -1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_3 + 2R_2 \to R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 2 & 1 \end{array}\right] \xrightarrow{-R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2 & -1 \end{array}\right]$$

So $B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -2 & -1 \end{bmatrix}$.

Also, $C^{-1} = -1 \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$.

$$A = B^{-1} D C^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -2 \\ -6 & 7 \end{bmatrix}.$$

Eksamen kont 2023

Oppgave 2

Regn ut Re z og Im z når

$$z = i^{(\sqrt{2}+i\sqrt{2})^4} - 3e^{\frac{\pi}{2}i}$$

Suppose that $z = a+ib$. Then in polar form

$z = re^{i\theta}$, where

$r = \sqrt{a^2+b^2}$ and $\tan\theta = \frac{b}{a}$.

$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$.

Consider $\sqrt{2}+i\sqrt{2}$. Then $r = \sqrt{(\sqrt{2})^2+(\sqrt{2})^2}$
$= \sqrt{2+2} = 2$ and

$\theta = \tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \pi/4$.

$\Rightarrow \sqrt{2}+i\sqrt{2} = 2e^{i\pi/4}$ and $(\sqrt{2}+i\sqrt{2})^4 = 2^4 e^{i\pi}$
$= 16(\cos\pi + i\sin\pi)$
$= -16$

$$\begin{cases} i^2 = -1 \\ i^4 = 1 \end{cases}$$

Next, $e^{\pi/2\, i} = \cos\pi/2 + i\sin\pi/2 = i$.

$z = i^{(\sqrt{2}+i\sqrt{2})^4} - 3e^{\pi/2\, i}$

$= i^{-16} - 3i$

$= \frac{1}{i^{16}} - 3i = 1 - 3i$

$$\begin{cases} \text{Re } z = 1 \\ \text{Im } z = -3 \end{cases}$$

Oppgave 3

La

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & -9 \\ 2 & -1 & 8 \end{bmatrix}$$

- Finn en basis for Col A og Null A.
- Har systemet
$$A\boldsymbol{x} = \boldsymbol{b}$$
Løsning for enhver $\boldsymbol{b} \in \mathbb{R}^3$? Hvis det har en løsning for en spesifikk $\boldsymbol{b}$, er denne løsningen da unik? Begrunn svaret.

We consider the row Echelon form of A:

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & -9 \\ 2 & -1 & 8 \end{bmatrix} \xrightarrow[R_3 - 2R_1 \to R_3]{R_2 + R_1 \to R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & -10 \\ 0 & -5 & 10 \end{bmatrix} \xrightarrow{R_3 + R_2 \to R_3}$$

Definisjon

Kolonnerommet til en reell $m \times n$-matrise

$$A = [\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \dots \quad \boldsymbol{a}_n]$$

er underrommet av $\mathbb{R}^m$ utspent av kolonnene i A:

$$\mathrm{C}(A) = \mathrm{span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n\}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & -10 \\ 0 & 0 & 0 \end{bmatrix}. \qquad \mathrm{Col}\,A = \mathrm{Sp}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \right\}.$$

So a basis for the column space of A, col A is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \right\}.$$

$$\text{Nul } A = \{ \vec{x} \in \mathbb{R}^3 \mid A\vec{x} = \vec{0} \}.$$

Next, we find the reduced row Echelon of A:

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & -10 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{5}R_2 \to R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 2R_2 \to R_1} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

So column 3 is not a pivot column. Let $x_3 = s$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3s \\ 2s \\ s \end{bmatrix} = s \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

$\text{Nul } A = \text{Sp} \left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\}$. Hence, $\left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul A.

Consider the system $A\vec{x} = \vec{b} \quad \forall\, \vec{b} \in \mathbb{R}^3$.

Since $\dim \text{col } A = 2 < 3$ and $\text{col } A \neq \mathbb{R}^3$.

So the system $A\vec{x}=\vec{b}$ does not have a solution for all $\vec{b}\in\mathbb{R}^3$. for example, $\begin{bmatrix}0\\0\\1\end{bmatrix}\in\mathbb{R}^3$ and $\nexists\ \vec{x}$: $A\vec{x}=\begin{bmatrix}0\\0\\1\end{bmatrix}$.

Since $\mathrm{Nul}\,A\neq\{\vec{0}\}$, if $A\vec{x}=\vec{b}$ has a solution for a specific $\vec{b}\in\mathbb{R}^3$, then $\vec{x}=\vec{x}_0+\vec{x}_p$. Here, $\vec{x}_0\in\mathrm{Nul}\,A$ and $\vec{x}_p$ is a particular solution of $A\vec{x}=\vec{b}$. $\exists$ infinitely many solutions $\vec{x}_0$ to $A\vec{x}=\vec{0}$. Hence, the solution is NOT unique!

Oppgave 5

- Finn egenverdiene til

$$A=\begin{bmatrix}2 & -1\\ 1/2 & 1/2\end{bmatrix}$$

Når det er kjent at $\begin{bmatrix}1\\1\end{bmatrix}$ og $\begin{bmatrix}2\\1\end{bmatrix}$ er egenvektorer.

An eigenvector $\vec{v}$ of a matrix A is a nonzero vector such that $A\vec{v}=\lambda\vec{v}$. λ is called an eigenvalue of A corresponding to $\vec{v}$.

$$\begin{bmatrix}2 & -1\\ 1/2 & 1/2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$ $\Leftrightarrow$ So the eigenvalue is 1.

$$\begin{bmatrix}2 & -1\\ 1/2 & 1/2\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}3\\1+1/2\end{bmatrix}=\begin{bmatrix}3\\3/2\end{bmatrix}=3/2\begin{bmatrix}2\\1\end{bmatrix}.$$

The eigenvalue is $3/2$.

Eksamen kont 2023

Oppgave 8

Benytt minste kvadraters metode til å finne andregradspolynomet

$$p(x) = ax^2 + bx + c$$

Som minimerer avstanden til datapunktene

x	-1	0	1	2
y	1	-2	0	5

(Det vil si, finn a, b og c).

Consider $p(x) = y$.

$$p(-1) = 1 \iff a - b + c = 1$$

$$p(0) = -2 \iff c = -2$$

$$p(1) = 0 \iff a + b + c = 0$$

$$p(2) = 5 \iff 4a + 2b + c = 5$$

The matrix equation is given by

$$\underbrace{\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ -2 \\ 0 \\ 5 \end{bmatrix}}_{\vec{b}}$$

Consider the normal equations $A^T A \vec{x} = A^T \vec{b}$ and solve for $\vec{x}$.

$$A^T A = \begin{bmatrix} 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 8 & 6 \\ 8 & 6 & 2 \\ 6 & 2 & 4 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 21 \\ 9 \\ 4 \end{bmatrix}$$

Consider the augmented matrix

$$\left[\begin{array}{ccc|c} 18 & 8 & 6 & 21 \\ 8 & 6 & 2 & 9 \\ 6 & 2 & 4 & 4 \end{array}\right] \xrightarrow{\frac{1}{18}R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 4/9 & 1/3 & 7/6 \\ 8 & 6 & 2 & 9 \\ 6 & 2 & 4 & 4 \end{array}\right] \xrightarrow[R_3 - 6R_1 \to R_3]{R_2 - 8R_1 \to R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 4/9 & 1/3 & 7/6 \\ 0 & 22/9 & -2/3 & -2/6 \\ 0 & -6/9 & 6/3 & -18/6 \end{array}\right] \xrightarrow[\frac{9}{6}R_3 \to R_3]{\frac{9}{22}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 4/9 & 1/3 & 7/6 \\ 0 & 1 & -3/11 & -3/22 \\ 0 & -1 & 3 & -9/2 \end{array}\right] \xrightarrow[R_3 + R_2 \to R_3]{R_1 - 4/9 R_2 \to R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 15/33 & \frac{81}{66} \\ 0 & 1 & -3/11 & -3/22 \\ 0 & 0 & 30/11 & -\frac{102}{22} \end{array}\right] \xrightarrow{\frac{11}{30}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 5/11 & 27/22 \\ 0 & 1 & -3/11 & -3/22 \\ 0 & 0 & 1 & -17/10 \end{array}\right] \xrightarrow[R_2 + \frac{3}{11}R_3 \to R_2]{R_1 - \frac{5}{11}R_3 \to R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3/5 \\ 0 & 0 & 1 & -17/10 \end{array}\right]. \Rightarrow \begin{cases} a = 2 \\ b = -3/5 \\ c = -17/10 \end{cases}$$

$$\Rightarrow \quad p(x) = 2x^2 - \frac{3}{5}x - \frac{17}{10}$$

Eksamen vår 2023

Oppgave 9

La

$$P_2(\mathbb{R}) = \{p(x) = ax^2 + bx + c \text{ for koeffisienter } a, b, c \in \mathbb{R}\}$$

være vektorrommet av reelle polynomer av grad høyst 2.

- Vis at $T: P_2(\mathbb{R}) \to \mathbb{R}^3$ definert ved

$$T(p) = \begin{bmatrix} p(0) \\ p'(1) \\ p''(2) \end{bmatrix} \qquad \left(p'(x) = \frac{\mathrm{d}p}{\mathrm{d}x}(x), \qquad p''(x) = \frac{\mathrm{d}^2 p}{\mathrm{d}x^2}(x)\right)$$

er en lineærtransformasjon og

- bestem ker T.

We show that T is a linear transformation:

Let $p(x), q(x) \in P_2(\mathbb{R})$ and $c \in \mathbb{R}$.

- $$T(p+q) = \begin{bmatrix} (p+q)(0) \\ (p+q)'(1) \\ (p+q)''(2) \end{bmatrix} = \begin{bmatrix} p(0) + q(0) \\ p'(1) + q'(1) \\ p''(2) + q''(2) \end{bmatrix}$$

$$= \begin{bmatrix} p(0) \\ p'(1) \\ p''(2) \end{bmatrix} + \begin{bmatrix} q(0) \\ q'(1) \\ q''(2) \end{bmatrix}$$

$$\Rightarrow \quad T(p+q) = T(p) + T(q).$$

- $$T(cp) = \begin{bmatrix} (cp)(0) \\ (cp)'(1) \\ (cp)''(2) \end{bmatrix} = \begin{bmatrix} c\,p(0) \\ c\,p'(1) \\ c\,p''(2) \end{bmatrix} = c \begin{bmatrix} p(0) \\ p'(1) \\ p''(2) \end{bmatrix}$$

$\Rightarrow \quad \tilde{T}(cp) = c\,\tilde{T}(p)$. Hence $\tilde{T}$ is a linear transformation.

$$\text{Ker } \tilde{T} = \{ p \in P_2(\mathbb{R}) \mid \tilde{T}(p) = \vec{0} \}$$

Let $p(x) = ax^2 + bx + c$.

$$\tilde{T}(p) = \vec{0} \iff \begin{bmatrix} p(0) \\ p'(1) \\ p''(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \qquad \begin{aligned} p'(x) &= 2ax + b \\ p''(x) &= 2a \end{aligned}$$

$$\iff \begin{bmatrix} c \\ 2a+b \\ 2a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{cases} a = 0 \\ b = 0 \\ c = 0 \end{cases}$$

$$\text{Ker } \tilde{T} = \{0\}$$

ALTERNATIVE

Find the matrix representation A of $\tilde{T}$.

$$A = \begin{bmatrix} T(1) & T(x) & T(x^2) \end{bmatrix}.$$

$$\tilde{T}(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{T}(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \tilde{T}(x^2) = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

$$\text{Nul } A = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Eksamen kont 2023

Oppgave 9

La $M_{n\times n}(\mathbb{R})$ betegne vektorrommet av reelle $n \times n$-matriser og la

$$S_n = \{A \in M_{n\times n}(\mathbb{R}) \;:\; A^\top = A\} \qquad (A^\top \text{ er den transponerte av } A)$$

- Bestem en basis for S_n når $n = 2$.

$$S_2 = \{A \in M_{2\times 2}(\mathbb{R}) \;:\; A^T = A\}$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

If $A^T = A$, then $b = c$.

$$S_2 = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; b = c \right\}.$$

Every member of S_2 is of the form $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$.

$$\Rightarrow \begin{bmatrix} a & b \\ b & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$$

$$= a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

So the basis for S_2 is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$