Plenumsregning 13: Andre ordens differensiallikninger

Ekstraoppgaver

Oppgave 1-2 c)

En modifisert oppgave 😊

Gitt følgende andreordens differensiallikning

$$y'' + y = 0$$

i. Skriv om til et system av førsteordens differensiallikninger.

Let $x_1(t) = y(t)$

$x_2(t) = y'(t)$

$$x_1'(t) = y'(t) = x_2(t)$$

$$x_2'(t) = y''(t) = -y(t) = -x_1(t)$$

So we obtain

$$x_1'(t) = x_2(t)$$

$$x_2'(t) = -x_1(t)$$

$$\Rightarrow \underbrace{\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}}_{\vec{x}_1'(t)} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\vec{x}(t)}$$

ii. Finn en generell basis for løsningsrommet og skriv generell løsning.

En modifisert oppgave 😊

Teorem 14.2 (forkortet)

Løsningsmengden til en homogen, annenordens differensiallikning

$$y''(t) + a_1 y'(t) + a_0 y(t) = 0$$

er et to-dimensjonalt, reelt vektorrom utspent av to lineært uavhengige funksjoner. Vi har 3 ulike tilfeller:

1) $y_1(t) = e^{\lambda_1 t}$ og $y_2(t) = e^{\lambda_2 t}$ hvis $\lambda_1 \neq \lambda_2,\ \lambda_1, \lambda_2 \in \mathbb{R}$
2) $y_1(t) = e^{at}\cos(bt)$ og $y_2(t) = e^{at}\sin(bt)$ hvis $\lambda_1 = \mathrm{a} + \mathrm{bi},\ \lambda_2 = \bar{\lambda}_1,\ \lambda_1, \lambda_2 \in \mathbb{C}$
3) $y_1(t) = e^{\lambda t}$ og $y_2(t) = te^{\lambda t}$ hvis $\lambda \in \mathbb{R}$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \iff \lambda = \pm i$$

Let $z = a + ib$ and so $a = 0$ and $b = 1$.

The basis for the solution space is

$$\begin{cases} y_1(t) = e^{at}\cos bt = \cos t \\ y_2(t) = e^{at}\sin bt = \sin t \end{cases}$$

$$y(t) = c_1 \cos t + c_2 \sin t \ //$$

Oppgave 3 b)

Finn en partikulær løsning for

$$y'' + y = \cos(t)$$

Consider the homogeneous equation : $y''(t) + y(t) = 0$

Characteristic equation : $\lambda^2 + 1 = 0 \iff \begin{cases} \lambda_1 = i \\ \lambda_2 = -i \end{cases}$

$$y_h(t) = c_1 \cos t + c_2 \sin t$$

For en *inhomogen, annenordens differensiallikning*, dvs. en likning på formen

$$y''(t) + a_1 y(t) + a_0 y(t) = f(t),$$

kan vi finne en **partikulær løsning**, y_p, ved hjelp av følgende formel:

$$y_{p(t)} = y_2(t) \int \frac{y_1(t) f(t)}{y_1(t) y_2'(t) - y_2(t) y_1'(t)} dt - y_1(t) \int \frac{y_2(t) f(t)}{y_1(t) y_2'(t) - y_2(t) y_1'(t)} dt$$

$$y_p(t) = y_2(t) \int \frac{y_1(t) f(t)}{W(y_1, y_2)} dt - y_1(t) \int \frac{y_2(t) f(t)}{W(y_1, y_2)} dt$$

where $W(y_1, y_2) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t) y_2'(t) - y_2(t) y_1'(t).$

$y_1(t) = \sin t$ and $y_2(t) = \cos t$

$y_1'(t) = \cos t$ and $y_2'(t) = -\sin t$;

$$W(y_1, y_2) = \sin t(-\sin t) - \cos t \cos t$$
$$= -\sin^2 t - \cos^2 t$$
$$= -(\sin^2 t + \cos^2 t)$$
$$= -1$$

$$\boxed{\cos^2(t) + \sin^2(t) = 1}$$

$f(t) = \cos t$

$$\cos^2(t) = \frac{1}{2}(\cos(2t) + 1)$$

$$\sin(t)\cos(t) = \frac{1}{2}\sin(2t)$$

$$y_p(t) = \cos t \int \frac{\sin t \cos t}{-1} dt - \sin t \int \frac{\cos t \cos t}{-1} dt$$

$$= -\frac{1}{2} \cos t \int \sin 2t \, dt + \frac{1}{2} \sin t \int (\cos 2t + 1) \, dt$$

$$y_p(t) = -\frac{1}{2}\cos t\left(-\frac{1}{2}\cos 2t\right) + \frac{1}{2}\sin t\left(\frac{1}{2}\sin 2t + t\right)$$

$$= \frac{1}{4}\cos t\cos 2t + \frac{1}{4}\sin t\sin 2t + \frac{t}{2}\sin t$$

$$\boxed{\begin{gathered}\sin(2t) = 2\sin(t)\cos(t)\\ \cos(2t) = 1 - 2\sin^2(t)\end{gathered}}$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$y_p(t) = \frac{1}{4}\left(\cos 2t\cos t + \sin 2t\sin t\right) + \frac{t}{2}\sin t$$

$$= \frac{1}{4}\cos(2t-t) + \frac{t}{2}\sin t$$

$$= \frac{1}{4}\cos t + \frac{t}{2}\sin t \qquad \therefore\ y_p(t) = \frac{t}{2}\sin t$$

NB: $\cos t$ is a homogeneous solution.

Eksamen høst 2019

Oppgave 2

Finn løsningen til initialverdiproblemet

$$y'' - 3y' + 2y = e^{2t}, \qquad y(0) = y'(0) = 1$$

Step 1: find $y_h(t)$

Step 2: find $y_p(t)$ and $y(t) = y_p(t) + y_h(t)$ and apply the initial conditions.

Characteristic equation: $\lambda^2 - 3\lambda + 2 = 0 \iff (\lambda-1)(\lambda-2) = 0$

$$\begin{cases}\lambda_1 = 2\\ \lambda_2 = 1\end{cases}$$

$$y_h(t) = c_1 e^{2t} + c_2 e^t$$

Teorem 14.2 (forkortet)

Løsningsmengden til en homogen, annenordens differensiallikning

$$y''(t) + a_1 y'(t) + a_0 y(t) = 0$$

er et to-dimensjonalt, reelt vektorrom utspent av to lineært uavhengige funksjoner. Vi har 3 ulike tilfeller:

4) $y_1(t) = e^{\lambda_1 t}$ og $y_2(t) = e^{\lambda_2 t}$ hvis $\lambda_1 \neq \lambda_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$
5) $y_1(t) = e^{at}\cos(bt)$ og $y_2(t) = e^{at}\sin(bt)$ hvis $\lambda_1 = a + bi$, $\lambda_2 = \bar{\lambda}_1$, $\lambda_1, \lambda_2 \in \mathbb{C}$
6) $y_1(t) = e^{\lambda t}$ og $y_2(t) = te^{\lambda t}$ hvis $\lambda \in \mathbb{R}$

$$\boxed{y_{p(t)} = y_2(t)\int \frac{y_1(t)f(t)}{y_1(t)y_2'(t) - y_2(t)y_1'(t)}dt - y_1(t)\int \frac{y_2(t)f(t)}{y_1(t)y_2'(t) - y_2(t)y_1'(t)}dt}$$

Let $y_1(t) = e^{2t}$; $y_2(t) = e^t$

$y_1'(t) = 2e^{2t}$; $y_2'(t) = e^t$

$f(t) = e^{2t}$

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$
$$= e^{2t}e^t - e^t 2e^{2t}$$
$$= e^{3t} - 2e^{3t} = -e^{3t}$$

$$y_p(t) = e^t \int \frac{e^{2t}e^{2t}}{-e^{3t}}dt - e^{2t}\int \frac{e^t e^{2t}}{-e^{3t}}dt$$

$$= -e^t \int e^t\,dt + e^{2t}\int dt$$

$$= -e^t e^t + te^{2t}$$

$$y_p(t) = -e^{2t} + te^{2t}$$
$$= (t-1)e^{2t}$$

$$y(t) = y_h(t) + y_p(t)$$
$$= c_1 e^{2t} + c_2 e^t + te^{2t} - e^{2t}$$
$$= (c_1 - 1)e^{2t} + c_2 e^t + te^{2t}$$

Initial Conditions : $\begin{cases} y(0) = 1 \\ y'(0) = 1 \end{cases}$

$$y'(t) = 2(c_1 - 1)e^{2t} + c_2 e^t + e^{2t} + 2te^{2t}$$

$$y(0) = c_1 - 1 + c_2 = 1 \iff c_1 + c_2 = 2$$
$$y'(0) = 2(c_1 - 1) + c_2 + 1 = 1 \iff 2c_1 + c_2 = 2$$

So $c_1 = 0$ and $c_2 = 2$.

$$y(t) = 2e^t - e^{2t} + te^{2t}$$

Eksamen vår 2017

Oppgave 2 a)

Finn to lineært uavhengige løsninger av den homogene differensiallikningen

$$y'' + 2y' + 2y = 0$$

Characteristic Equation : $\lambda^2 + 2\lambda + 2 = 0$

$$\lambda = \frac{-2 \pm \sqrt{4 - 4(2)}}{2} = \frac{-2 \pm i2}{2}$$

So $\begin{cases} \lambda_1 = -1+i \\ \lambda_2 = -1-i \end{cases}$

Let $z = a+ib$. Then $a=-1$ and $b=1$. So

$y_1(t) = e^{-t}\cos t$ and $y_2(t) = e^{-t}\sin t$

Eksamen høst 2017

Oppgave 4

Likningen for en udempet tvungen harmonisk bevegelse er gitt ved

$$y''(t) + y(t) = \cos(t-2)$$

a) Finn den generelle løsningen til den homogene likningen.

$$y''(t) + y(t) = 0$$

Characteristic Equation: $\lambda^2 + 1 = 0 \iff \lambda = \pm i$ $\begin{cases} \lambda_1 = i \\ \lambda_2 = -i \end{cases}$

Hence, the homogeneous solution is

$$y_h(t) = c_1 \cos t + c_2 \sin t$$

b) Finn den generelle løsningen til den inhomogene likning

Let $f(t) = \cos(t-2)$,

$y_1(t) = \cos t$ and $y_2(t) = \sin t$.

$\Rightarrow$ $y_1'(t) = -\sin t$ and $y_2'(t) = \cos t$.

$$W(y_1, y_2) = y_1(t) y_2'(t) - y_2(t) y_1'(t)$$

$$= \cos t \cos t - \sin t(-\sin t)$$

$$= \cos^2 t + \sin^2 t = 1$$

$$y_p(t) = y_2(t)\int \frac{y_1(t) f(t)}{W(y_1,y_2)}\,dt - y_1(t)\int \frac{y_2(t) f(t)}{W(y_1,y_2)}\,dt$$

$$= \sin t \int \cos t \cos(t-2)\,dt - \cos t \int \sin t \cos(t-2)\,dt$$

Let $\tilde{I} = \int \cos t \cos(t-2)\,dt$ and $\tilde{II} = \int \sin t \cos(t-2)$

Consider $\cos(t-2) = \cos t \cos 2 + \sin t \sin 2$ - - - - - - - - (*)

(NB: from (*) it is clear that $\cos(t-2)$ is a homogeneous solution)

$$\tilde{I} = \int \cos t\left(\cos t \cos 2 + \sin t \sin 2\right) dt$$

$$= \cos 2 \int \cos^2 t\,dt + \sin 2 \int \cos t \sin t\,dt$$ NB: $\cos 2$ & $\sin 2$ are constants.

$$= \cos 2 \int \frac{1}{2}(\cos 2t + 1)\,dt + \sin 2 \int \frac{1}{2}\sin 2t\,dt$$

$$= \cos 2\left(\frac{1}{4}\sin 2t + \frac{1}{2}t\right) + \sin 2\left(-\frac{1}{4}\cos 2t\right)$$

$$= \frac{1}{4}\left(\sin 2t \cos 2 - \cos 2t \sin 2\right) + \frac{1}{2}t\cos 2$$

$$\tilde{I} = \frac{1}{4}\sin(2t-2) + \frac{1}{2}t\cos 2 \;//$$

$$\tilde{II} = \int \sin t\left(\cos t \cos 2 + \sin t \sin 2\right) dt$$

$$= \cos 2 \int \sin t \cos t\,dt + \sin 2 \int \sin^2 t\,dt$$

$$= \cos 2 \int \frac{1}{2} \sin 2t \, dt + \sin 2 \int \frac{1}{2} (1 - \cos 2t) \, dt$$

$$= -\frac{1}{4} \cos 2t \cos 2 - \frac{1}{4} \sin 2t \sin 2 + \frac{1}{2} t \sin t$$

$$= -\frac{1}{4} \cos(2t-2) + \frac{1}{2} t \sin t \,//$$

Now,

$$y_p(t) = \sin t \; I - \cos t \; II$$

$$= \frac{1}{4} \sin t \sin(2t-2) + \frac{t}{2} \sin t \cos 2 + \frac{1}{4} \cos t \cos(2t-2) - \frac{t}{2} \cos t \sin 2$$

$$= \frac{1}{4} \left[\cos(2t-2) \cos t + \sin(2t-2) \sin t \right] + \frac{t}{2} \left[\sin t \cos 2 - \cos t \sin 2 \right]$$

$$= \frac{1}{4} \cos(t-2) + \frac{t}{2} \sin(t-2)$$

from $(*)$ we know that $\cos(t-2)$ is a homogeneous solution.

So, $y_p(t) = \frac{t}{2} \sin(t-2)$.

$$y(t) = y_h(t) + y_p(t)$$

$$= c_1 \cos t + c_2 \sin t + \frac{t}{2} \sin(t-2).$$