TMA4110 Matematikk 3

Brief solutions to Assignment 6

Chapter 11

Exercise 1:

a) We have that

$$\mathbf{v}^T A\mathbf{v} = \mathbf{v}^T \lambda\mathbf{v} = \lambda\langle \mathbf{v}, \mathbf{v}\rangle_e.$$

where $\langle -, -\rangle_e$ denote the standard inner product. This is positive since $\lambda > 0$

b) Since A is symmetric $\mathbb{R}^n$ admits a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of orthogonal eigenvectors of A. It follows that we can write

$$\mathbf{u} = a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n.$$

Where $a_j \neq 0$ for some j. Thus, we get

$$\begin{aligned}\mathbf{u}^T A\mathbf{u} &= \mathbf{u}^T A\left(\sum_{i=1}^n a_i\mathbf{v}_i\right) \\ &= \sum_{i=1}^n a_i\mathbf{u}^T A(\mathbf{v}_i) \\ &= \sum_{i=1}^n a_i\mathbf{u}^T \lambda_i\mathbf{v}_i\end{aligned}$$

where λ_i is the eigenvalue of $\mathbf{v}_i$. Now, we have that

$$\begin{aligned}\mathbf{u}^T \lambda_i\mathbf{v}_i &= \sum_{j=1}^n a_j\mathbf{v}_j^T \lambda_i\mathbf{v}_i \\ &= \sum_{j=1}^n a_j\lambda_i\mathbf{v}_j^T\mathbf{v}_i \\ &= a_i\lambda_i.\end{aligned}$$

It follows that

$$\begin{aligned}\mathbf{u}^T A\mathbf{u} &= \sum_{i=1}^n a_i\mathbf{u}^T \lambda_i\mathbf{v}_i \\ &= \sum_{i=1}^n a_i^2\lambda_i.\end{aligned}$$

Now since $a_i \neq 0$ for some i and $\lambda > 0$, we have that $\mathbf{u}^T A\mathbf{u} > 0$.

c) We have shown positivity above. So, symmetry and linearity remains. For symmetry, we have

$$\begin{aligned}\langle \mathbf{u}, \mathbf{w}\rangle &= \mathbf{u}^T A\mathbf{w} \\ &= (A^T\mathbf{u})^T\mathbf{w} \\ &= (\mathbf{w}^T(A^T\mathbf{u}))^T \\ &= \mathbf{w}^T A^T\mathbf{u} \\ &= \mathbf{w}^T A\mathbf{u} \\ &= \langle \mathbf{w}, \mathbf{u}\rangle.\end{aligned}$$

Thus it suffices to prove linearity in the 2nd variable.

$$\begin{aligned}\langle \mathbf{u}, \mathbf{w}_1 + \mathbf{w}_2\rangle &= \mathbf{u}^T A(\mathbf{w}_1 + \mathbf{w}_2) \\ &= \mathbf{u}^T A\mathbf{w}_1 + \mathbf{u}^T A\mathbf{w}_2 \\ &= \langle \mathbf{u}, \mathbf{w}_1\rangle + \langle \mathbf{u}, \mathbf{w}_2\rangle.\end{aligned}$$

Chapter 12

Exercise 2: This amounts to solving the least-squares system $A^\top A\boldsymbol{x} = A^\top\boldsymbol{b}$, whose general solution is

$$\boldsymbol{x} = \begin{bmatrix}4\\ s\\ 2\\ 0\end{bmatrix}, \qquad \text{where } s \in \mathbb{R} \text{ is a free parameter.}$$

Exercise 3: Solve $A^\top A\boldsymbol{x} = A^\top\boldsymbol{b}$, noting that the systems are written as $[A \mid \boldsymbol{b}]$.

Exercise 4: We seek a polynomial $p(x) = ax^2 + bx + c$ that best fits the given points. Let $\boldsymbol{x} = (a, b, c)$ be the unknown coefficients. Then the constraints are that

$$\begin{aligned}0 &= p(0) = 0a + 0b + c,\\ 1 &= p(-1) = a - b + c,\\ 1 &= p(1) = a + b + c,\\ 2 &= p(2) = 4a + 2b + c\end{aligned}$$

or $A\boldsymbol{x} = \boldsymbol{b}$ in matrix form, with

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

Now solve $A^\top A\boldsymbol{x} = A^\top \boldsymbol{b}$ to get $\boldsymbol{x} = (0.5, -0.1, 0.3)$, and thus $p(3) = 4.5$.

Exercise 5: Stochastic matrix: A square matrix for which all entries should be nonnegative and columns should sum to 1. This excludes **a)** and **c)**. This is because the matrix in **a)** is not a square matrix and the sum of the entries in the second column of the matrix in **c)** is greater than 1.

The matrix in **b)** is regularly stochastic since all entries in M^3 are strictly positive.

The unique stationary state or equilibrium state is the eigenvector $\begin{bmatrix} 5/19 \\ 10/19 \\ 4/19 \end{bmatrix}$ associated to the eigenvalue 1, whose entries sum to 1.

Exercise 6: The eigenvectors associated with the eigenvalue 1 are on the form

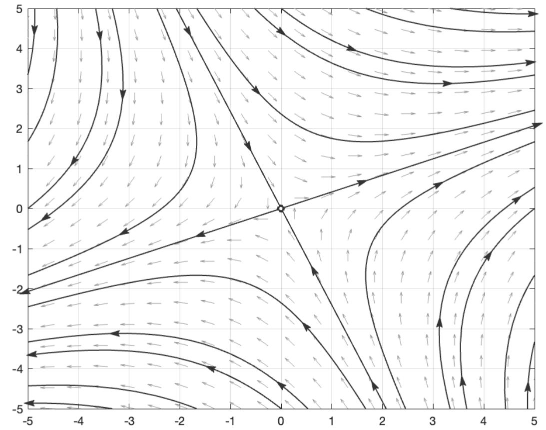
$$\begin{bmatrix} b \\ a \end{bmatrix} t, \qquad t \in \mathbb{R},$$

so the unique eigenvector whose entries sum to 1 is

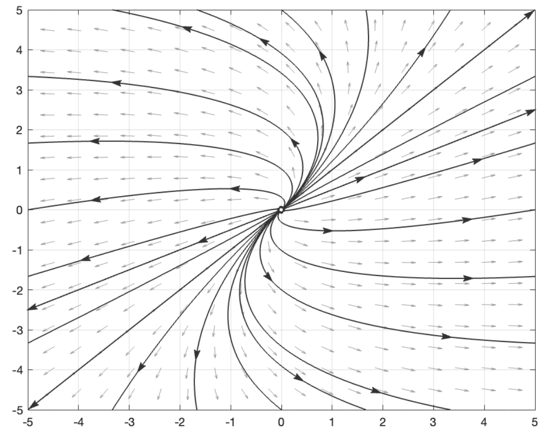
$$\frac{1}{a+b} \begin{bmatrix} b \\ a \end{bmatrix}.$$

Exercise 7: The number answers provide the general solutions. Taking into account the initial conditions one finds that for a) one gets $c_1 = 3$, $c_2 = -1$ and $c_3 = 0$, while for b) one gets $c_1 = \frac{1}{2}$, $c_2 = \frac{4}{3}$ and $c_3 = -\frac{1}{6}$.

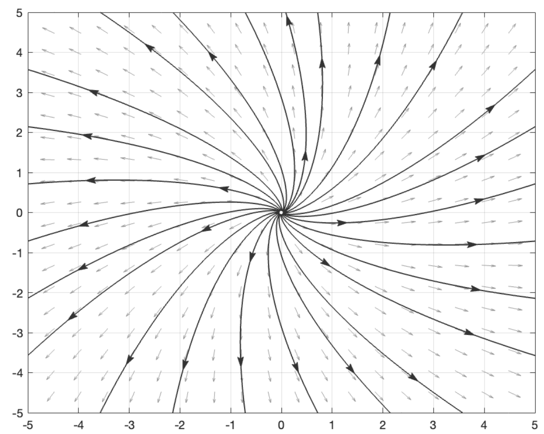
Exercise 8: a). Eigenvalues are -3 and 2.



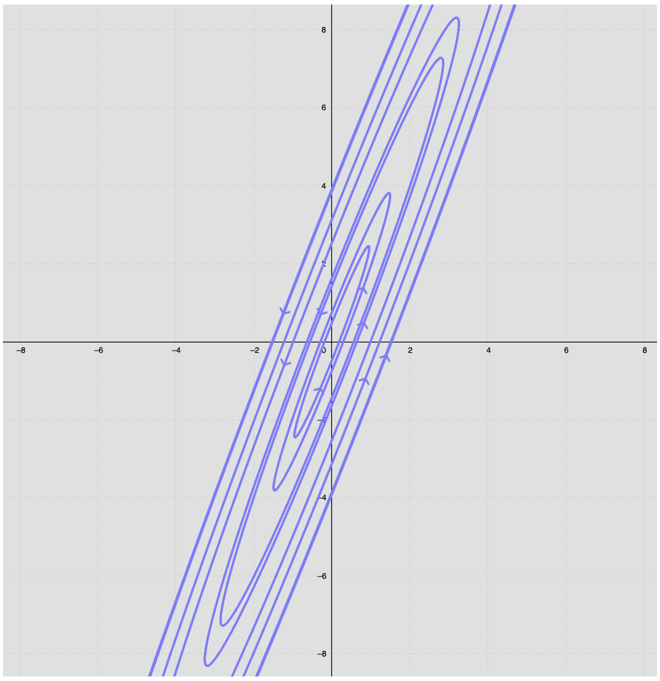
b). Eigenvalues are 3 and 4.



c). Eigenvalues are 3+i3 and 3-i3.



d). Eigenvalues are -2i and 2i.



Exercise 9: a). Eigenvalues are put on the diagonal of D, for instance in the order:

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Put the associated eigenvectors in the same order as columns in matrix P:

$$P = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

b). Let $\boldsymbol{y} = (y_1, y_2, y_3)$. Then the general solution is

$$\boldsymbol{y}(t) = c_1 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The specific solution satisfying the initial conditons becomes

$$\boldsymbol{y}(t) = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix} e^{-t} - \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} e^{-2t} + \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$

c). $\lim_{t \to \infty} \boldsymbol{y}(t) = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$