TMA4110 Matematikk 3

Brief solutions to Assignment 5

Chapter 10

Exercise 1: Evaluate if the formula $Ax = \lambda x$ holds for some λ.

(a) Yes. With eigenvalue 1.

(b) Yes. With eigenvalue -1.

(c) Yes. With eigenvalue 2.

(d) No. The two vectors are not on the same line.

(e) Yes. The eigvenvalue is 0.

Exercise 2: Use the characteristic polynomial to determine eigenvalues λ_i. Then solve the equation $(A - \lambda_i I)x = \mathbf{0}$ for each eigenvalue. For matrix b) we compute the characteristic polynomial as $\det(A - \lambda I_3)$. To get

$$\chi_A = \lambda^2(2 - \lambda)$$

So the eigenvalues are 0 and 2.

Exercise 3: For 2 a) we have 2 eigenvalues of a 2x2 matrix. Hence, the algebraic multiplicity of each is 1. Use that geometric multiplicity is smaller equal the algebraic multiplicity and at least 1.
In 2 b) we got the eigenvalues 0 and 2 which have algebraic multiplicity 2 and 1 respectively. We now determine the eigenspaces, for $\lambda = 0$. So we compute the null space of $A - 0I_3$.

$$A - 0I_3 = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

From here, it is easy to see that the eigenspace is spanned by $(0, 1, 0)^T$. Which has geometric multiplicity 1. Similarly, for $\lambda = 2$ we get

$$A - 3I_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ -1 & 0 & -2 \end{pmatrix}$$

So we get that the eigenspace is spanned by $(4, 1, -2)^T$. Which also has geometric multiplicity 1.

Exercise 4:

(a) Yes, 0 is an eigenvalue if and only if $\dim \operatorname{Null} A > 0$. In this case the $\operatorname{Null} A = E_0$, where E_0 is the eigenspace of 0. Also, since the geometric multiplicity of 0 is 3, the algebraic multiplicity of 0 is $\geqslant 3$.

  (i) If A has two distinct eigenvalues, then the possible values for its algebraic multiplicity are 3, 4, and 5.

  (ii) If A has four distinct eigenvalues, then the only possible value for its algebraic multiplicity is 3.

(b) In 4.a)(i), None of the values for the algebraic multiplicity of 0 can make A to be necessarily diagonalizable.

In 4.a)(ii), A is always diagonalizable.

Exercise 5: No they can not, as the eigenvalues have to come in conjugate pairs and $6 + i$ is not among the values.

Exercise 6:

(a) Let A be a matrix such that A^2 is the zero matrix $\mathbf{0}$. Let λ be an eigenvalue of A. Then there exists an eigenvector $\mathbf{v} \neq 0$ such that $A\mathbf{v} = \lambda\mathbf{v}$. If $A = \mathbf{0}$, then $\lambda = 0$ since $\mathbf{v}$ is nonzero.

Now, we assume that $A \neq \mathbf{0}$. Then by multiplying A on both sides of the equation $A\mathbf{v} = \lambda\mathbf{v}$, we get $A^2\mathbf{v} = \lambda A\mathbf{v}$. So, $\lambda A\mathbf{v} = 0$ and $\lambda = 0$ since $A \neq \mathbf{0}$ and $\mathbf{v}$ is nonzero.

(b) There are infinitely many examples here, for example, choose $A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

(c) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Its characteristic equation is given by

$$\lambda^2 - (a + d)\lambda + ad - bc = 0.$$

If 0 is the only eigenvalue of A, then we get the two equations: $a+d=0$ and $ad-bc=0$. From the two equations,

$$A^2 = \begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(d) $\det(\mathbf{0}) = \det(A^2) = (\det(A))^2 = 0$. Hence, $\det A = 0$ and A is not invertible. Apply Theorem 10.4.

Chapter 11

Exercise 7: A matrix is diagonalizable if and only if there is a basis of eigenvectors. It follows that 2 a) is diagonalizable and 2 b) not.

Exercise 8: If you swap the order of the columns in P this is exactly what would see (note that if x is an eigenvector so is $-x$).

Exercise 9: Determine eigenvalues and eigenvectors as usual. Check algebraic and geometric multiplicities. It follows that a) and c) are diagonalizable and b) not.

Exercise 10:

(a) Determine eigenvectors as usual for the columns of matrix P and the corresponding eigenvalues for the D.

(b) Use that $B^{2101} = PD^{2101}P^{-1}$ and that D is very simple.

Exercise 11:

(a) To determine the matrix we evaluate T on the elements of the basis $(1, x, x^2)$. $T(1) = -1$, $T(x) = -x$ and $T(x^2) = 2(2x^2+1) - x^2 = 3x^2 + 2$. So the matrix is

$$A = \begin{pmatrix} -1 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

with eigenvalues -1 and 3.

(b) For $\lambda = -1$: the eigenvectors are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

For $\lambda = 3$: the eigenvector is $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

The matrix A is diagonalizable because the set of eigenvectors $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\}$ spans $\mathbb{R}^3$.

Exercise 12:

(a) Just do as you usually would and then normalize in the end.

(b) Same procedure as you are used to.

(c) Compute the inner products to see they are orthogonal for all possible eigenvectors.

(d) It follows from the above that P is an orthogonal matrix, it follows that $P^TP = I_2$ by the argument from last homework sheet. It follows from uniqueness of inverses that $P^T = P^{-1}$.