TMA4110 Matematikk 3

Brief solutions to Assignment 3

Chapter 7

Exercise 1: Make it clear to your self what you have to prove, and remember that addition and scalar multiplication is done point-wise.

The difference is that one is the zero vector, that is an element of V and the other is the zero scalar. One example is $0 \in \mathbb{R}$ and $(0,0)^T \in \mathbb{R}^2$.

Exercise 2:

a) Yes. This is the line $y = -x$ going through origo.

b) No. this set does not contain the zero vector.

c) No. This set is not closed under scalar multiplication with real scalars.

Exercise 3: Make it clear to your self what you have to prove, and remember that addition and scalar multiplication is done point-wise.

Let T_n denote the set of lower triangular $n \times n$-matrices. We have to show 3 things. That the zero vector is in T_n, that if $v, u \in T_n$ then $u + v \in T_n$ and if $a \in \mathbb{C}$ and $u \in T_n$ then $au \in T_n$. The zero vector in M_n is the 0 matrix, this is lower triangular, so $0 \in T_n$. If u and v are lower triangular, then $(u+v)_{ij} = u_{ij} + v_{ij}$, so it follows that $u + v$ is lower triangular. Closure under scalar multiplication is done similarly.

Exercise 4: Check if $Av = 0$ and use Gauss elimination to determine the basis for Col$A$.

The dimension of Col$A$ is the number of pivot elements of the reduced row echelon form of A.

Exercise 5: We solve the question for A

$$A = \begin{bmatrix} 5 & -3 & 2 & 21 & -3 \\ 0 & 1 & -2 & 3 & 1 \\ 1 & 0 & -1 & -7 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 58 & 4 \\ 0 & 1 & 0 & 133 & 11 \\ 0 & 0 & 1 & 65 & 5 \end{bmatrix}.$$

So we have that

$$\left\{ \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} \right\}$$

is a basis for Col$A$. We also have that

$$\left\{ \begin{pmatrix} -58 \\ -133 \\ -65 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ -11 \\ -5 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for Null$A$. Finally, since every row had a pivot element we get that the rows of A form a basis for Row$A$.

Exercise 6:

a) We can for instance choose $(1, x, x^2)$. A basis is – per definition – a list of vectors which span out the space and is linearly independent. To span out: Any arbitrary vector $a + bx + cx^2$ is a linear combination $a \cdot 1 + b \cdot x + c \cdot x^2$ av 1, x og x^2. Linearly independent: Given an equation $a \cdot 1 + b \cdot x + c \cdot x^2 = 0$, we must show that we only have the trivial solution. But a polynomial of degree 2 can maximally have two zeros, hence the equation cannot hold unless $a = 0$, $b = 0$ og $c = 0$; i.e., we only have the trivial solution.

b) In the basis coordinates chosein in **a)**, a polynomial $a_0 + a_1 x + a_2 x^2$ corresponds to the vector $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$. In particular, $1 + 2x + 3x^3$ is written $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

c) The standrad basis for $\mathcal{P}_n$ is $\{1, x, x^2, \cdots, x^n\}$.

Exercise 7:

a) Since A has 4 rows, then Col$A \leqslant 4$.

b) By rank-nullity we have that

$$5 = \text{Null}A + \dim(\text{Col}A) = 2 + \dim(\text{Col}A)$$

which implies that $\dim(\text{Col}A) = 3$. It follows that there must be 3 linearly independent columns.

c) If A is non-zero, then there exists some v with $Av \neq 0$. It follows that $\dim \mathrm{Col} A \geqslant 1$. Now by **Theorem 7.26** we have that $\dim \mathrm{Col} A = \dim \mathrm{Row} A$ so there must be a least one linearly independent.

d) We have that $\dim \mathrm{Row}(A^T) = \dim \mathrm{Col}(A)$, so by rank-nullity we have that $\dim \mathrm{Col}(A) = 2$.

e) No since $\dim \mathrm{Col} A \leqslant 4$ we have by rank-nullity that $\mathrm{Null} A \geqslant 1$.

Exercise 8:

a) The two are equivalent. It is clear that if U is a subspace then $cu + v \in U$ for $u, v \in U$ and $c \in \mathbb{R}$. So we assume that the statement in the exercise holds. Note that if we set $u = v$ and $c = -1$, then $cu + v = -u + u = 0$ so $0 \in U$. Furthermore, if we set $c = 1$ then $u + v \in U$ if $u, v \in U$.

b) Yes. Since U is non-empty then there exists an $x \in U$, so by the above we have that $-x + x \in U$, but $-x + x = 0$.

Chapter 8

Exercise 9:

a) We have to find $a, b \in \mathbb{R}$ such that

$$a\begin{pmatrix} 1 \\ 2 \end{pmatrix} + b\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

Solving the linear system we get that

$$\left[\begin{array}{cc|c} 1 & -1 & 5 \\ 2 & 3 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & \frac{17}{5} \\ 0 & 1 & -\frac{8}{5} \end{array}\right].$$

so setting $a = \frac{17}{5}$ and $b = -\frac{8}{5}$ we get what we wanted. Now by linearity

$$\begin{aligned} T\left(\begin{pmatrix} 5 \\ 2 \end{pmatrix}\right) &= T\left(\frac{17}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{8}{5}\begin{pmatrix} -1 \\ 3 \end{pmatrix}\right) \\ &= \frac{17}{5}T\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) - \frac{8}{5}T\left(\begin{pmatrix} -1 \\ 3 \end{pmatrix}\right) \\ &= \frac{17}{5} \cdot \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} - \frac{8}{5}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 17 \\ \frac{26}{5} \\ -5 \end{pmatrix}. \end{aligned}$$

b) This is the same procedure as a), but with the vectors e_1, e_2.

c) T cannot be surjective since $\dim \mathbb{R}^2 < \dim \mathbb{R}^3$. On the other hand we do have that T is injective. To see this note that $\dim \operatorname{im} T = 2$, since the vectors given in the description of the exercise are linearly independent. Now by rank-nullity this implies $\dim \ker T = 0$, so $\ker T = \{0\}$, which is equivalent to T being injective.

Exercise 10: For the standard matrix note that the standard matrix of a composite is the product of the matrices. To determine the kernel of $R \circ T$ note that R is injective so $\ker(R \circ T) = \ker T$. Now $3x_1 + 2x_2 = 0$ if and only if $-3/2x_1 = x_2$. So the kernel is exactly the vectors in $\mathbb{R}^2$ satisfying this relationship.

Exercise 11: We consider the matrix

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 1 & -2 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$

to see that the base-change matrix must be

$$\begin{bmatrix} 2 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Exercise 12: We argue by the conclusions of exercise 8.a) and 8.b). Let U be a subspace of V and $T\colon V \to W$ a linear map. Since $0 \in U$ we have that $0 = T(0) \in T(U)$. So $T(U)$ is non-empty. Now suppose $u, v \in T(U)$ and $c \in \mathbb{R}$, then there exists $u', v' \in U$ such that $T(u') = u$ and $T(v') = v$, now since U is a subspace we have that $cu' + v' \in U$, so $T(cu' + v') \in T(U)$. Now, using linearity we have that:

$$T(cu' + v') = cT(u') + T(v') = cu + v.$$

So $T(U)$ is a subspace of W.

Exercise 13:

a) To check if a function $T : \mathcal{P} \to \mathcal{P}$ is linear, we must check two conditions: i) $T(p_1(x) + p_2(x)) = T(p_1(x)) + T(p_2(x))$ for all polynomials $p_1(x)$ og $p_2(x)$, and ii) $T(c \cdot p(x)) = cT(p(x))$ for all polynomials $p(x)$ and scalars c.
A polynomial may be written $p(x) = a_0 + a_1 x + \cdots + a_n x^n$. The derivative of p is $D(p(x)) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + na_n x^{n-1}$. You may explicitly check that this formula (satisfies i) og ii)).
G: i)

$$\begin{aligned} G(p_1(x) + p_2(x)) &= x \cdot (p_1(x) + p_2(x)) \\ &= x \cdot p_1(x) + x \cdot p_2(x) \\ &= G(p_1(x)) + G(p_2(x)). \end{aligned}$$

ii)

$$G(c \cdot p(x)) = x \cdot (c \cdot p(x)) = c \cdot (x \cdot p(x)) = c \cdot G(p(x)).$$

b) The image of D is all polynomials which may be written as the derivative of another polynomial. Given a polynomial

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

, we see that

$$P(x) = a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots + \frac{a_n}{n+1} x^{n+1}$$

is the antiderivative of p; $P' = p$. But this means that $p = D(P)$. Thus, the image of D is $\mathcal{P}$; all polynomials can be written as the polynomial of the derivative of another polynomical.
The kernel of D is all polynomials which are sent to zero, i.e., all polynomial whose derivative is zero. These are the constant polynomials. The kernel of D are all the constant polynomials.
The image of G are all polynomials $p(x)$ written as $p(x) = G(q(x))$ for some polynomial $q(x)$. That is, $p(x) = xq(x)$. The images is thus all polynomials with x as a factor, or – equivalently – minimum one zero in $x = 0$.
The kernel of G are alle polynomials $p(x)$ for which $G(p(x)) = x \cdot p(x) = 0$. This is only the zero-polynomial.

c) Remember that a linear transformation $T : V \to W$ is injective if and only if the kernel only consists of the zero-vector; surjectivity if and only if the image is W .
From **b)** it follows that D is surjective, but not injective; G is injective, not surjective.

d) If we use the product rule for derivation (Matte 1), we see that

$$(x \cdot p(x))' = x' \cdot p(x) + x \cdot p'(x) = p(x) + x \cdot p'(x).$$

Thus:

$$D(G(p(x))) = p(x) + G(D(p(x)))$$

which means

$$(D \circ G)(p) - (G \circ D)(p) = p$$

for any polynomial p, hence

$$(D \circ G) - (G \circ D) = \mathrm{id}_{\mathcal{P}}$$

e) Let $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_2$ an $\mathbf{e}_3$ be polynomials given by:

$$\mathbf{e}_0(x) = 1 \qquad \mathbf{e}_2(x) = x^2$$
$$\mathbf{e}_1(x) = x \qquad \mathbf{e}_3(x) = x^3$$

Then $(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2)$ is a basis for $\mathcal{P}_2$, and $(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ a basis for $\mathcal{P}_3$.
With respect to these bases we get the following matrices:

$$\text{The matrix for } D_3\text{:} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\text{The matrix for } G_3\text{:} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$