EXTRA TASKS

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\-2\\1\end{bmatrix},\ T\left(\begin{bmatrix}-1\\2\\0\end{bmatrix}\right) = \begin{bmatrix}1\\4\\1\end{bmatrix} \text{ and } T\left(\begin{bmatrix}1\\3\\-1\end{bmatrix}\right) = \begin{bmatrix}3\\1\\5\end{bmatrix}.$$

i. Find the standard matrix A corresponding to the transformation T.

ii. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the columns of A. Show that the columns of A form an orthogonal set and find the orthogonal projection of the vector $\mathbf{u} = \begin{bmatrix}2\\1\\3\end{bmatrix}$ to $\text{Span}\{\mathbf{v}_1, \mathbf{v}_3\}$.

2. Show in each of the following cases whether or not the subsets of the given vector spaces are subspaces.

i. $U_1 = \left\{ \begin{bmatrix}a\\b\end{bmatrix} \;\middle|\; 2a - 3b = 0 \text{ and } a, b \in \mathbb{R} \right\} \subseteq \mathbb{R}^2$

ii. $U_2 = \left\{ \begin{bmatrix}a & 0\\c & d\end{bmatrix} \;\middle|\; ad = 0 \text{ and } a, c, d \in \mathbb{R} \right\} \subseteq M_{2\times 2}$, where $M_{2\times 2}$ is the vector space of all 2×2 matrices.

iii. $U_3^m = \left\{ \begin{bmatrix}6s - 4t\\2s + t\\t - m\end{bmatrix} \;\middle|\; s, t \in \mathbb{R} \right\} \subseteq \mathbb{R}^3$

a. when $m = 0$ and

b. when $m = 1$.

3. Let $C[0, \pi/2]$ be the vector space of continuous functions defined on $[0, \pi/2]$ with inner product

$$\langle f, g\rangle = \int_0^{\frac{\pi}{2}} f(t)g(t)\ dt,$$

where $f, g \in C[0, \pi/2]$. Let $W = \text{Span}\{1, \sin t\}$.

i. Is the set $\{1, \sin t\}$ an orthogonal set? Justify your answer.

ii. Find an orthogonal basis for W.

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - 3x_2 \\ -6x_1 + 9x_2 \end{bmatrix}.$$

i. Is T injective?

ii. Is T surjective?

5. Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + 2x_3 \\ 2x_1 + x_2 + x_3 \end{bmatrix}.$$

i. Is S injective?

ii. Is S surjective?

6. Find the general solution of the system

$$\begin{array}{cccccc} (i-1)x_1 & -4x_2 & & = & 4 \\ ix_1 & -2ix_2 & +(2i+1)x_3 & = & i+1 \\ (3i+1)x_1 & & +(2i-1)x_3 & = & 5. \end{array}$$