Problem 1 Find all solutions to the linear system

$$\begin{cases} x - y - 3z = 2 \\ -x + 2y + 5z = -1 \\ 3x + 3y + 3z = 12 \end{cases}$$

Problem 2 • Solve $z^4 = (1 - \mathrm{i})^{12}$ for $z \in \mathbb{C}$ and • sketch the root(s) in the complex plane.

Problem 3 Find the general solution to

$$y''(t) - 4y'(t) + 4y(t) = t^2 + 1.$$

Problem 4 • Calculate a basis for both Null A and Col A when

$$A = \begin{bmatrix} 1 & -2 & 1 & 8 \\ -1 & 2 & 1 & -2 \\ 1 & -2 & -2 & -1 \end{bmatrix},$$

and • determine the dimension of $\mathrm{Null}(A^\top)$, where $A^\top$ is the transpose matrix of A.

Problem 5 • Find the eigenvalues of

$$A = \begin{bmatrix} 2 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

when it is known that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ are eigenvectors.

• Then solve the initial-value problem

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = 2x_1 - x_2 \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = \dfrac{1}{2}x_1 + \dfrac{1}{2}x_2 \end{cases} \qquad x_1(0) = 1, \qquad x_2(0) = -1.$$

Problem 6 Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation for which

$$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\1\end{bmatrix}.$$

Determine $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$.

Problem 7 Let $V \subseteq \mathbb{R}^3$ be the linear span $\operatorname{Sp}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}\right\}$.

- Compute the orthogonal projection of $\boldsymbol{b} = \begin{bmatrix}1\\0\\2\end{bmatrix}$ onto V.
- Find the least-squares solution to

$$\begin{cases} x_1 + x_2 = 1 \\ x_1 + x_2 = 0 \\ x_1 - x_2 = 2 \end{cases}$$

Problem 8 Find the matrix A which satisfies the equation

$$\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 2 & -2 & -1\end{bmatrix} A \begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 1\\ -1 & 1\\ 1 & 3\end{bmatrix}.$$

Problem 9 Let

$$\mathcal{P}_2(\mathbb{R}) = \left\{p\colon \mathbb{R} \to \mathbb{R} \,\middle|\, p(x) = ax^2 + bx + c \text{ for coefficients } a, b, c \in \mathbb{R}\right\}$$

be the vector space of real polynomials of degree at most 2.

- Show that $T\colon \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3$ defined by

$$T(p) = \begin{bmatrix}p(0)\\ p'(1)\\ p''(2)\end{bmatrix} \qquad \left(p'(x) = \frac{\mathrm{d}p}{\mathrm{d}x}(x), \quad p''(x) = \frac{\mathrm{d}^2p}{\mathrm{d}x^2}(x)\right)$$

is a linear transformation and • determine $\ker T$.

Problem 10 Let A be a column vector (that is, an $n \times 1$ matrix) and let B be a matrix such that the product AB is a well-defined square matrix.

- Determine the number of rows and columns in B and AB.
- Find the rank of AB assuming that AB is a nonzero matrix.