Problem 1 • Compute a basis for the column space and the null space of the matrix

$$A = \begin{bmatrix} 5 & -15 & -1 & 12 \\ -2 & 6 & 0 & -4 \\ 4 & -12 & -2 & 12 \end{bmatrix}.$$

• Also determine $\dim \operatorname{Null}(A^\top)$.

Problem 2 Find the solution of the system

$$\begin{aligned} x_1' &= x_1 + 2x_2 \\ x_2' &= 4x_1 + 3x_2 \end{aligned}$$

that satisfies the conditions $x_1(0) = 1$ and $x_2(0) = 5$.

Problem 3 Determine all the complex solutions of the equation

$$z^3 - 3z^2 + 6z - 4 = 0.$$

Write the solutions in polar form. *Hint: Maybe you can directly see one solution?*

Problem 4 • Compute the determinant of

$$A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix}.$$

• What is the number of solutions to the system of equations

$$A\boldsymbol{x} = \begin{bmatrix} 6 \\ -12 \\ 13 \\ 5 \end{bmatrix} ?$$

(Justify your answer.)

Problem 5 • Determine whether the polynomials

$$p(t) = 2 - t, \qquad q(t) = 1 + t^2 \qquad \text{and} \qquad r(t) = 1 + 2t - 3t^2$$

are linearly independent or not in $\mathcal{P}_2(\mathbb{R})$, the vector space of real second-degree polynomials.

• Then determine whether p and q are orthogonal with respect to the inner product

$$\langle f, g \rangle = \sum_{k=0}^{2} f(k)g(k), \qquad f, g \in \mathcal{P}_2(\mathbb{R}).$$

Problem 6 Find the general solution to the differential equation

$$y'' + y' - 6y = 5e^{2t} - 36t.$$

Problem 7 An orthogonal basis for the column space of the matrix

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$$

is given by $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$, where

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

- Compute the projection of $\boldsymbol{b} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ onto Col A. (The answer is $\begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$.)
- Find the least-squares solution to the overdetermined system $A\boldsymbol{x} = \boldsymbol{b}$.

Problem 8 Two different ordered bases for $\mathbb{R}^2$ are $\mathscr{B} = (\boldsymbol{b}_1, \boldsymbol{b}_2)$ and $\mathscr{C} = (\boldsymbol{c}_1, \boldsymbol{c}_2)$, where

$$\boldsymbol{b}_1 = \begin{bmatrix} -1 \\ 8 \end{bmatrix}, \quad \boldsymbol{b}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \quad \boldsymbol{c}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \text{and} \quad \boldsymbol{c}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

- A vector has coordinates $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with respect to basis $\mathscr{B}$. What are the coordinates of this vector in the standard basis?
- Determine the change-of-basis matrix from $\mathscr{C}$ to $\mathscr{B}$.

Problem 9 Let $T \colon V \to W$ be a linear transformation between two vector spaces V and W, and let U be a subspace of V. Show that

$$T(U) = \{T(u) \mid u \in U\}$$

is a subspace of W.

Problem 10 Let $A \neq 0$ be an $n \times n$ matrix for which $A^3 = 0$. Show that $I - A$ is an invertible matrix. *Hint: A matrix B is invertible if there exists a matrix C such that $BC = I = CB$.*