Norwegian University of Science and Technology
Department of Mathematical Sciences

Contact during the exam: Eldar Straume
(Telephone 735 96683)

EXAM IN MA1201 LINEAR ALGEBRA AND GEOMETRY

Monday 3rd December 2007
Time: kl. 09.00 - 13.00
Permitted aids: No permitted aids.
English

All answers must be justified. All problems will count the same when grading the exam.

Grades: 21st December 2007.

Problem 1

a) We consider the following system of equations

$$\begin{array}{rcrcrcl} x & + & 2y & - & 3z & = & 1 \\ 2x & + & y & + & 3\alpha z & = & 2 \\ 2x & + & & & 2z & = & \beta \end{array}$$

where α and β are constants. For what values of α and β does the system of equations have

(i) no solutions (ii) infinite number of solutions (iii) exactly one solution?

b) Find the reduced echelon form of the matrix

$$A = \begin{bmatrix} 1 & 2 & -3 & 1 \\ 2 & 1 & 3 & 2 \end{bmatrix}$$

and find a parametrization of the straight line which is the intersection of the two planes given by the equations

$$\begin{array}{rcrcrcl} x & + & 2y & - & 3z & = & 1 \\ 2x & + & y & + & 3z & = & 2. \end{array}$$

Problem 2 Find all numbers z in the complex plane such that $z^3 = 1 - \sqrt{3}i$. Write the solution(s) on polar form and illustrate these on a figure.

Problem 3 Let $T_A \colon \mathbb{R}^2 \to \mathbb{R}^3$ and $T_B \colon \mathbb{R}^3 \to \mathbb{R}^2$ be two linear transformations with standard matrices equal to respectively

$$A = \begin{bmatrix} 1 & -3 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 & 4 \\ -2 & 4 & -8 \end{bmatrix}.$$

a) Calculate $T_A(x, y)$ and $T_B(x, y, z)$. What is $T_A(4, -5)$?

b) Decide whether T_A is **one-to-one (injective)**. Find the standard matrix of the composition of the two transformations that maps $\mathbb{R}^2$ to $\mathbb{R}^2$. Decide whether or not this transformation is invertible.

Problem 4

a) Let the matrix A be given as $A = \begin{bmatrix} 11 & 5\sqrt{3} \\ 5\sqrt{3} & 1 \end{bmatrix}$. Show that A has eigenvalues equal to -4 and 16.

b) Find an orthogonal matrix P and a diagonal matrix D such that $P^{-1}AP = D$ where A is the matrix given in **a)**.

Let P be the standard matrix for the linear transformation $T_P \colon \mathbb{R}^2 \to \mathbb{R}^2$. What is the geometric interpretation of this linear transformation?

c) Consider the conic section $11x^2 + 10\sqrt{3}xy + y^2 = 25$.

Introduce a new coordinate system such that the equation for the conic section is expressed in standard form. Sketch the conic section in the xy-coordinate system, but the position of the new coordinate system should be clearly marked.

Problem 5

a) Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ where a, b and c are arbitrary real numbers. Show that A has only real eigenvalues.

b) Let B be an $(n \times n)$-matrix (n a positive integer). Assume that the real number λ is an eigenvalue for B. Show that λ^2 is an eigenvalue for B^2. Give a counterexample that shows that the opposite implication is not generally true (in other words λ^2 an eigenvalue for B^2 does not always imply that $\pm\lambda$ is an eigenvalue for B).

(Hint: Find a (2×2)-matrix with a known geometric interpretation.)