Norwegian University of Science and Technology
Department of Mathematical Sciences

Contact during the exam:
Harald Krogstad 73 59 35 36 and Dag Madsen 73 59 66 82

EXAM IN COURSE TMA4110 Calculus 3

English
Tuesday November 30, 2004
Hours: 9-13
Aids: C. (Approved calculator, Rottmann: *Matematisk formelsamling, English)*
Grading finished: December 21, 2004

It should be clearly stated how all answers are obtained.

Problem 1

Find all complex numbers z such that

$$z^3 = 1 + \sqrt{3}\,i.$$

Write the solutions in polar form, $re^{i\theta}$. Sketch the solutions in the complex plane.

Problem 2

(a) Solve the initial value problem $y'' + 9y = 0$, $y(0) = 1$, $y'(0) = 6$.

(b) Find the general solution to the equation $y'' + 9y = 6e^{3x} + \sin 3x$.

(c) Find the general solution to the equation $x^2y'' + 2xy' - 6y = 0, x > 0$.

Problem 3

Let

$$A = \begin{bmatrix} 2 & 1 & 2 & 1 & 4 \\ 1 & 1 & 0 & 1 & 1 \\ -1 & 0 & -2 & 0 & -3 \\ 0 & -1 & 2 & -1 & 5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 4 \end{bmatrix}.$$

(a) Solve the system of equations $A\mathbf{x} = \mathbf{b}$ by bringing the total matrix (extended coefficient matrix) over to reduced Echelon form.

(b) Find a basis for $\text{Row}\,(A)$, $\text{Col}\,(A)$, and $\text{Row}\,(A)^{\perp}$. State the dimension of each of these vector spaces.

Problem 4

Let A be the matrix

$$A = \begin{bmatrix} -2 & -2 & 4 \\ -4 & 0 & 4 \\ -4 & -2 & 6 \end{bmatrix}.$$

(a) Show that the eigenvalues of A are $\lambda = 0$ and $\lambda = 2$. Find a basis for each eigenspace.

(b) If possible, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

(c) Solve the following system of differential equations

$$\begin{aligned} y_1' &= -2y_1 - 2y_2 + 4y_3, \\ y_2' &= -4y_1 + 4y_3, \\ y_3' &= -4y_1 - 2y_2 + 6y_3, \end{aligned}$$

when $y_1(0) = 0,\ y_2(0) = 3,\ y_3(0) = 1$.

Problem 5

Lake A in Bymarka is to be treated with rothenon poison by pouring M kilos of rothenon into the lake at $t = 0$. A river is flowing from A into lake B. Lake A has a water volume V and B a volume $3V$. The amount of water (per time unit, m^3/s) out from A is U, whereas the amount of water out from B is $6U$. We assume that the mixture in lake A and lake B is uniform and that the volumes of A and B are constant.

(a) Show that the amounts of rothenon in A, $y_1(t)$, and B, $y_2(t)$, satisfy the following system of differential equations:

$$\begin{aligned} \frac{dy_1(t)}{dt} &= -U\frac{y_1(t)}{V}, \\ \frac{dy_2(t)}{dt} &= U\frac{y_1(t)}{V} - 2U\frac{y_2(t)}{V}, \quad t \geq 0. \end{aligned}$$

What are the initial conditions for y_1 and y_2?

(b) Find the solution of the system in **(a)** when $U/V = 1$ and $M = 1$.

Problem 6

In a 3×3-matrix the sum of the elements in each row is equal to 4. Show that such a matrix always has an eigenvector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. What is the corresponding eigenvalue?

Norwegian University of Science and Technology
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EXAM IN TMA4110 CALCULUS 3

English
Wednesday December 20, 2006
Time: 9–13

You may use the following (code C): Approved calculator (HP30S)
Rottman: *Matematisk formelsamling*

Grades to be announced: January 19, 2007

All answers have to be justified (with the exception of Problem 3). When grading, the 12 problems (1, 2abc, 3, 4, 5abc, 6ab, 7) will, as a rule, have the same weight.

Problem 1 Solve the equation

$$z^3 = \frac{4}{1 - i\sqrt{3}}.$$

Write the solutions in the form $re^{i\theta}$. Use a figure to show the location of the solutions in the complex plane.

Problem 2 Find a general solution of the differential equations

a) $y'' - y' + 2.5y = 0,$

b) $y'' + y' - 2y = x + e^{-2x},$

c) $4x^2y'' + 8xy' - 3y = 0, \quad x > 0.$

Problem 3 *Multiple choice problem — to be answered without explanations by choosing one alternative for each question.*

Let $y_1(x)$ and $y_2(x)$ be two solutions of the constant coeffisient equation $y'' + ay' + by = 0$. If the Wronskian $W(y_1, y_2)$ equals 0 when $x = 0$, what is the value of $W(y_1, y_2)$ when $x = 1$?

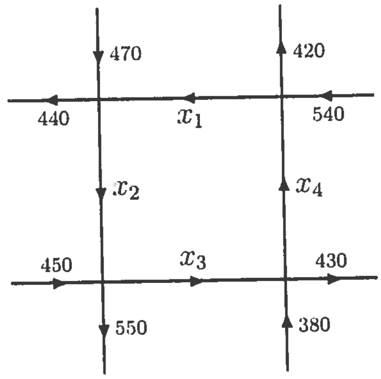
A: 0 **B:** 1 **C:** e **D:** The value depends on a and b.

Which one of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ is in the null space Null(A) of the matrix

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 0 & 1 & -3 \\ 2 & 1 & 1 \end{bmatrix}?$$

A: $\mathbf{v}_1 = (2, 1, 1)$ **B:** $\mathbf{v}_2 = (2, -3, 1)$ **C:** $\mathbf{v}_3 = (-2, 3, 1)$ **D:** $\mathbf{v}_4 = (-1, 2, 0)$

Problem 4 Four one-way streets in a city intesect as shown in the figure to the right. The number of cars passing per hour is shown in the figure.



Write down a system of equations in the form $A\mathbf{x} = \mathbf{b}$ which $\mathbf{x} = (x_1, x_2, x_3, x_4)$ must satisfy, and solve the system.

Find x_1, x_2 and x_3 if the x_4-section is closed for traffic, making $x_4 = 0$.

Problem 5 A matrix A and a vector $\mathbf{b}$ is given by

$$A = \begin{bmatrix} 1 & 1 & -2 & 0 & -1 \\ 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

a) Find a basis for the row space Row(A) and a basis for the column space Col(A).

b) What is the dimension of Row(A), Col(A) and the orthogonal complement Row$(A)^\perp$?

c) Show that the orthogonal complement Col$(A)^\perp$ is spanned by the vector $\mathbf{u} = (3, -1, 1, 3)$. Find, for example by using $\mathbf{u}$, a condition which b_1, b_2, b_3 and b_4 must satify in order that the system $A\mathbf{x} = \mathbf{b}$ shall be consistent.

Problem 6

a) Compute the eigenvalues of the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}.$$

Find an orthogonal matrix P with determinant equal to 1 such that $P^T AP$ is a diagonal matrix.

b) A conic section has equation

$$3x^2 + 4xy - 1 = 0.$$

Introduce a new, rotated coordinate system such that the conic section is in standard position relative to the new coordinate system. Make a figure to show the location of the conic section and the new coordinate axes in the xy-plane.

Problem 7 Let A be an $n \times n$-matrix.

Show that if 5 is an eigenvalue of A, then 3 is an eigenvalue of $A - 2I$ where I denotes the $n \times n$ identity matrix.

Also show that if $\mathbf{v}$ is an n-vector satisfying $A\mathbf{v} \neq \mathbf{0}$ and $A^2\mathbf{v} = \mathbf{0}$, then $\mathbf{v}$ and $A\mathbf{v}$ are linear independant.

Norwegian University of Science and Technology
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EXAM IN TMA4110 CALCULUS 3

English
Monday, December 3, 2007
9 am – 1 pm

You may use the following (code C): Approved calculator (HP30S)
Rottman: *Matematisk formelsamling*

Grades to be announced: January 3, 2008

All answers have to be justified. When grading, the 12 problems (1, 2abc, 3, 4, 5, 6abc, 7ab) will, as a rule, have the same weight.

Problem 1

Write down the polar form of the complex number $iz(\bar{z})^{-1}$, where $z = re^{i\theta}$ and $z \neq 0$.
Show on a figure all solutions of the equation $iz = \bar{z}$.

Problem 2

a) The equation

$$y'' + ay' + by = 0$$

with constant coefficients has solutions $y_1 = e^{(3/2)x}$ and $y_2 = e^{-(1/2)x}$. Determine a and b.

b) Solve the initial value problem

$$y'' - 4y' + 5y = 0, \qquad y(0) = 1,\ y'(0) = 4.$$

c) Find a general solution of the equation

$$y'' - 2y' - 15y = e^{5x} - 17\cos 3x.$$

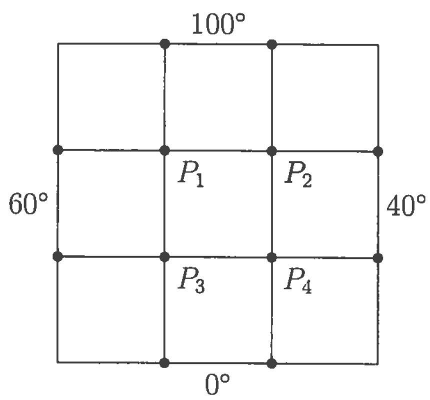
Problem 3 The equation

$$y'' + 2(\tan x)y' - y = 0$$

has a solution $y_1(x) = \sin x$. Find another solution y_2 of the form $y_2(x) = u(x)y_1(x)$ where u is a non-constant function.

The following integral is given: $\displaystyle\int \frac{\cos^2 x}{\sin^2 x}\,dx = -\frac{\cos x}{\sin x} - x + C.$

Problem 4 We shall determine the temperatures T_1, T_2, T_3, T_4 at four points P_1, P_2, P_3, P_4 on the square plate, see the figure on the right. Temperatures along the sides of the plate are shown on the figure.



Set up and solve a system of equations for T_1, T_2, T_3, T_4 if we assume that the temperature at each point P_1 to P_4 is the average of the temperatures at the four neighboring points (to the right, above, to the left, and below).

Problem 5 Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent vectors in $\mathbb{R}^n$. Are vectors

$$\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2, \quad \mathbf{w}_2 = \mathbf{v}_1 + \mathbf{v}_3, \quad \mathbf{w}_3 = \mathbf{v}_2 + \mathbf{v}_3$$

linearly dependent or linearly independent? (Remember to explain the answer.)

Problem 6

a) Find a basis for the solution space of the homogeneous system

$$\begin{aligned} x_1 - 2x_2 + x_3 \phantom{{}+2x_4} &= 0 \\ x_1 \phantom{{}-2x_2} - x_3 + 2x_4 &= 0. \end{aligned}$$

b) Determine the orthogonal projection of the vector $(1, 2, -3, 1)$ onto the subspace $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ of $\mathbb{R}^4$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal vectors given by

$$\mathbf{v}_1 = (1, -2, 1, 0), \quad \mathbf{v}_2 = (1, 0, -1, 2).$$

c) Find also vectors $\mathbf{v}_3$ and $\mathbf{v}_4$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is an orthogonal basis for $\mathbb{R}^4$.

Problem 7 The matrix A is given by

$$A = \begin{bmatrix} 10 & -9 \\ 6 & -5 \end{bmatrix}.$$

a) Find the eigenvalues and eigenvectors of A, and write down a general solution of the system of differential equations

$$\begin{aligned} y_1' &= 10y_1 - 9y_2 \\ y_2' &= \ \ 6y_1 - 5y_2. \end{aligned}$$

b) Determine an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Find a matrix B such that $B^2 = A$.

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EXAM IN TMA4110/4115 CALCULUS 3

English
Saturday, August 18, 2007
9 am – 1 pm

You may use the following (code C): Simple calculator (HP30S), with user's manual
Rottman: *Matematisk formelsamling*

Grades to be announced: September 10, 2007

All answers have to be justified. When grading, the 12 problems (1, 2abc, 3ab, 4ab, 5ab, 6, 7) will, as a rule, have the same weight.

Problem 1 Use polar form $z = re^{i\theta}$ to find the solutions of the equation

$$z^3 - 5\bar{z} = 0.$$

Problem 2

a) Solve the initial value problem

$$y'' - 2y' + 5y = 0, \quad y(0) = 0, \quad y'(0) = 4.$$

b) Find a general solution of the differential equation

$$y'' + y' - 12y = 7e^{-4x}.$$

c) Find a particular solution of the differential equation

$$y'' - \frac{2}{x^2}y = x^2, \qquad x > 0.$$

Problem 3

a) Solve the system

$$\begin{aligned} x_1 + x_2 + 3x_3 + 2x_4 &= 0 \\ 3x_1 - 2x_2 - x_3 - 4x_4 &= 0 \\ 4x_1 + x_2 + 6x_3 + 2x_4 &= 0. \end{aligned}$$

b) Let the 3×4 matrix A be given by

$$A = \begin{bmatrix} 1 & 1 & 3 & 2 \\ 3 & -2 & -1 & -4 \\ 4 & 1 & 6 & 2 \end{bmatrix}.$$

Find a basis for each of the spaces $\mathrm{Row}(A)$, $\mathrm{Col}(A)$, $\mathrm{Null}(A)$ and $\mathrm{Col}(A)^{\perp}$.

Problem 4

a) A square 3×3 matrix A is given by

$$A = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 1 & 0 & a \end{bmatrix}.$$

For which real numbers a is the matrix A invertible?

b) Find A^{-1} when $a = 1$.

Problem 5

a) Find the eigenvalues and the eigenvectors of the matrix

$$A = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}.$$

Write down an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

b) In a metropolitan area with a constant total population, 7 million people are now living in the city and 5 millions are living in the suburbs. Each year 20 % of the people in the city move to the suburbs (and 80 % remain in the city), while 10 % of the people living in the suburbs move into the city (and 90 % remain in the suburbs).

Find the long-term distribution of the population between the city and its suburbs.

Problem 6 It is given that 1 and 3 are the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

Let $x_1 = x_1(t)$, $x_2 = x_2(t)$ and $x_3 = x_3(t)$ be differentiable functions of t. Solve the system of differential equations

$$\begin{aligned} x_1' &= 2x_1 - x_2 + x_3 \\ x_2' &= -x_1 + 2x_2 + x_3 \\ x_3' &= 3x_3 \end{aligned}$$

with initial conditions $x_1(0) = 1$, $x_2(0) = 2$, $x_3(0) = 1$.

Problem 7 Let A be an $m \times n$ matrix and B an $n \times p$ matrix such that $AB = 0$. Explain why $\mathrm{Col}(B)$ is contained in $\mathrm{Null}(A)$, and show that

$$\mathrm{rank}(A) + \mathrm{rank}(B) \leq n.$$

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EKSAMEN I TMA4115 MATEMATIKK 3

Engelsk
Onsdag 2. juni 2004
Kl. 9–14

Hjelpemidler (kode C): Enkel kalkulator (HP30S), med tilhørende bruksanvisning
Rottman: *Matematisk formelsamling*

Sensurdato: 23. juni

Give reasons for all answers (with the exception of Problem 4).

Problem 1 *Complex numbers*

a) Write the complex number

$$w = \left(-\frac{\sqrt{3}}{4} + \frac{i}{4}\right)^3$$

in the form $re^{i\theta}$. Find all complex numbers z such that $z^3 = w$. Write the answer in the form $a + ib$ where a and b are real numbers. Use exact values for a and b.

b) Let z be an arbitrary complex number. A square $OABC$ in the complex plane has one corner in the origin O, and the corners are given counter clockwise. If the corner A is the number z, what are the corners B and C expressed by z?

Problem 2 *First order differential equations*

a) Solve the initial value problem

$$y' + \frac{2}{x}y = \frac{\cos x}{x^2}, \qquad y(\pi/2) = 0.$$

b) Use Euler's method with step size $h = 0.5$ to find approximates $y_1 \approx y(2.5)$ and $y_2 \approx y(3.0)$ to the solution $y(x)$ of the initial value problem

$$y' = 1 + (x - y)^2, \qquad y(2) = 1.$$

Problem 3 *Second order differential equations*

a) Find a general solution of the differential equation

$$y'' + y' - 2y = e^x + e^{2x}.$$

b) The differential equation

$$(*) \qquad (1 - x^2)y'' + 2xy' - 2y = 0, \qquad -1 < x < 1$$

has two solutions of the form $y_1 = x + a$ and $y_2 = x^2 + b$ where a and b are constants. Find a and b by substitution. Explain why the solutions y_1 and y_2 are linearly independent, and give general solution of $(*)$.

c) Find a particular solution of the differential equations

$$(1 - x^2)y'' + 2xy' - 2y = 6\left(1 - x^2\right)^2, \qquad -1 < x < 1.$$

Problem 4 *Multiple choice problem - to be answered without explanations.*

a) Let A and B be $2 \times 2-$ matrices. If the determinant of A is 2 and the determinant of B is 3, what is the determinant of $C = -2A^{-1}B^T$?

A: 3 **B:** -3 **C:** 6 **D:** -6

b) For which (n) c are the vectors $\mathbf{v}_1 = (1, 3, -3)$, $\mathbf{v}_2 = (-2, 4, 1)$, $\mathbf{v}_3 = (-1, 1, c)$ linearly independent?

A: No c **B:** $c = 1$ **C:** $c \neq 1$ **D:** All c

Problem 5 *Matrices and systems of linear equations*
Given the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 2 & -2 & 3 \\ 2 & 4 & -3 & 5 \end{bmatrix}.$$

a) Solve the system of equations $A\mathbf{x} = \mathbf{0}$.

b) Find a basis for the row space $\mathrm{Row}(A)$ and for the column space $\mathrm{Col}(A)$.

c) Show that the vector $\mathbf{y} = (1, 5, -3)$ lies in $\mathrm{Col}(A)^\perp$. What other vectors does $\mathrm{Col}(A)^\perp$ consist of? Give reason for your answer.

Problem 6 *Eigenvalues and eigenvectors*

a) Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 4 & 0 & 4 \\ 1 & -1 & 4 \end{bmatrix}.$$

Show that $\mathbf{v}_1 = (-1, 3, 1)$ and $\mathbf{v}_2 = (0, 2, 1)$ are eigenvalues of A by computing $A\mathbf{v}_1$ and $A\mathbf{v}_2$.

b) Find an eigenvector $\mathbf{v}_3$ of A such that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent. Write up an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

c) Let $y_1 = y_1(t)$, $y_2 = y_2(t)$ and $y_3 = y_3(t)$ be differentiable functions of t. Solve the system of differential equations

$$\begin{aligned} y_1' &= y_1 + y_2 - 2y_3 \\ y_2' &= 4y_1 \qquad\quad + 4y_3 \\ y_3' &= y_1 - y_2 + 4y_3 \end{aligned}$$

with initial conditions $y_1(0) = 0$, $y_2(0) = 1$ and $y_3(0) = 2$.

Problem 7 *Symmetric matrices*
Show in general that if *all* eigenvalues of a symmetric $n \times n$ – matrix A are positive $(\lambda > 0)$, then $\mathbf{x}^T A\mathbf{x} > 0$ for all vectors $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$.

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EXAM IN TMA4115 CALCULUS 3

English
Monday, August 9, 2004
Time: 0900–1400

Permitted aids (Code C): Approved calculator: (HP 30S),
Karl Rottmann: Matematisk Formelsamling.

Grades to be announced: September 1.

Justify all your answers.

Problem 1 Find all complex solutions of the equation

$$z^2 + (3+3i)z + 5i = 0.$$

Write the solutions on the form $z = x + iy$, and mark the solutions in the complex plane.

Problem 2 Find all real numbers x, y so that the complex number $z = x + iy$ satisfies

$$|z + i| = |z - 1|.$$

Sketch the solution set in the complex plane.

Problem 3 Explain why $y' = x^2 + y^2$ not is a linear first order differential equation. Use Euler's method with step length $h = 0.1$ to approximate $y(0.2)$ when

$$y' = x^2 + y^2, \qquad y(0) = 1.$$

Problem 4 Find a linear homogen second order differential equation $y'' + ay' + by = 0$ with general solution

$$y = e^x \left(c_1 e^{x\sqrt{2}} + c_2 e^{-x\sqrt{2}}\right).$$

Problem 5 Solve the differential equations

a) $y'' + y = \cos x$

b) $y'' + y = 1/\cos x$ $\qquad -\frac{\pi}{2} < x < \frac{\pi}{2}.$

Problem 6 A tank contains to begin with 100 liter pure water. Then a brine containing 50 gram of salt per liter starts running into the tank at a rate of 2 liter per minute. The diluted brine leaves the tank at the same rate.

After 10 minutes the brine running into the tank is replaced by pure water. (The rate of the liquid in and out of the tank is as before 2 liter per minute.)

a) How much salt is there in the tank after 10 minutes?

b) How much salt is there in the tank after 20 minutes?

The salt mixture in the tank is kept uniform by stirring.

Problem 7

a) Solve the system of equations

$$\begin{aligned} x + 2y \qquad &= 1 \\ 3x + 7y + 2z &= 2 \\ 2x + 3y - 3z &= 0. \end{aligned}$$

b) Decide for which values of the parameters a and b the system of equations

$$\begin{aligned} x + 2y \quad &= 1 \\ 3x + 7y + 2z &= b \\ 2x + 3y - az &= 0 \end{aligned}$$

has

i) exactly one solution

ii) infinitely two solutions

iii) infinitely many solutions

iv) no solutions.

Problem 8 If the null space Null(A) of a 5×6-matrix A has dimension 4, what can you say about the dimension of the column space Col(A) and the row space Row(A)?

Problem 9 Given vectors $\mathbf{v}_1 = [2, 1, 0, 0]$, $\mathbf{v}_2 = [3, 0, 1, 0]$, $\mathbf{v}_3 = [4, 1, 0, 1]$ and $\mathbf{b} = [0, 0, 0, 9]$. If $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, what is then the orthogonal projection of $\mathbf{b}$ onto

a) V?

b) $V^{\perp}$?

Problem 10

a) Find the eigenvalues and eigenvectors of the matrix
$$A = \begin{bmatrix} 7 & 1 \\ 1 & 7 \end{bmatrix}.$$

b) Make a shift of variables $\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} x' \\ y' \end{bmatrix}$ that brings the quadratic form

$$7x^2 + 2xy + 7y^2$$

into a quadratic form without mixed terms ($x'y'$). Write down the matrix P and the new quadratic form.

Problem 11 Let A be a diagonalizable matrix where every eigenvalue is either 1 or -1. Show that $A^{-1} = A$.

Norwegian University of Science and Technology
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EXAM IN TMA4115 CALCULUS 3

English
Wednesday, June 7, 2006
Time: 9–13

You may use the following (code C): Approved calculator (HP30S)
Rottman: *Matematisk formelsamling*

Grades to be announced: June 28.

All answers have to be justified (with the exception of Problem 4).

Problem 1 Calculate

$$w = \left(\sqrt{3} + i\right)^4.$$

Find the solution of the equation $z^4 = w$ lying in the *second* quadrant in the complex plane.

Problem 2

a) Solve the initial value problem

$$y'' + 6y' + 10y = 0, \quad y(0) = 0, \quad y'(0) = 2.$$

b) Find a general solution of the differential equation

$$y'' - 3y' + 2y = e^x + 10\sin x.$$

c) Suppose that p and q are functions such that the differential equation

$$y'' + p(x)y' + q(x)y = 0, \qquad x > 0$$

has a basis $y_1 = x^2$, $y_2 = x^2 \ln x$ of solutions. Find a general solution of the equation

$$y'' + p(x)y' + q(x)y = x, \qquad x > 0.$$

Problem 3 Given the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 2 & 4 & 3 & 3 & 1 \\ 1 & 2 & 2 & 1 & 1 \end{bmatrix}.$$

a) Solve the system of equations $A\mathbf{x} = \mathbf{0}$, and find a basis for $\mathrm{Null}(A)$.

b) Find a basis for each of the spaces $\mathrm{Col}(A)$, $\mathrm{Row}(A)$ and $\mathrm{Row}(A)^{\perp}$.

c) Show that the vector $\mathbf{v} = (2, 1, -1, 0)$ is in the orthogonal complement $\mathrm{Col}(A)^{\perp}$. Is $\{\mathbf{v}\}$ a basis for $\mathrm{Col}(A)^{\perp}$?

Problem 4 *Multiple choice problem — to be answered without explanations.*

a) Find the least squares solution $(\bar{x}, \bar{y})$ of the system

$$\begin{aligned} x + 3y &= 5 \\ x - y &= 1 \\ x + y &= 0. \end{aligned}$$

A: $(0, 1)$ **B:** $(1/2, 3/2)$ **C:** $(1, 1)$ **D:** $(3/2, 1/2)$

b) For which real numbers α, β is P an orthogonal matrix with determinant equal 1?

$$P = \frac{1}{2} \begin{bmatrix} \alpha & -1 & -1 & \beta \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ \alpha & 1 & 1 & \beta \end{bmatrix}$$

A: $\alpha = 2, \beta = 0$ **B:** $\alpha = 1, \beta = -1$ **C:** $\alpha = -1, \beta = 1$ **D:** $\alpha = 0, \beta = 2$

Problem 5 Given the matrix

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}.$$

a) Show that A has eigenvalues 0 and 1. Find all the eigenvectors of A.

b) Find an invertible matrix P and a diagonal matrix D such that

$$P^{-1}AP = D.$$

Solve the system of differential equations

$$\begin{aligned} y_1' &= 3y_1 - y_2 - 2y_3 \\ y_2' &= 2y_1 \qquad\; - 2y_3 \\ y_3' &= 2y_1 - y_2 - \;\; y_3 \end{aligned}$$

with initial conditions $y_1(0) = 0$, $y_2(0) = 1$ og $y_3(0) = 2$.

c) A square matrix B is said to be *idempotent* if $B^2 = B$. Show that the matrix A is idempotent. Show in general that if λ is an eigenvalue of an idempotent matrix, then $\lambda = 0$ or $\lambda = 1$.

Norwegian University of Science and Technology
Department of Mathematical Sciences

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EXAM IN TMA4115 CALCULUS 3
English
Monday, May 21, 2007
9 am – 1 pm

You may use the following (code C): Simple calculator (HP30S) with user's manual
Rottman: *Matematisk formelsamling*

Grades to be announced: June 11, 2007

All answers have to be justified. When grading, the 12 problems (1, 2ab, 3ab, 4ab, 5ab, 6ab, 7) will, as a rule, have the same weight.

Problem 1

Show on a figure all complex numbers z such that

$$|z| = |z+1|.$$

Find the solutions of the equation

$$z^4 = (z+1)^4.$$

Problem 2

a) Solve the initial value problem

$$y'' + 2y' + y = 0, \quad y(0) = 2, \; y'(0) = 3.$$

b) Find a general solution of the differential equation

$$y'' + 2y' + y = x + 2e^{-x}.$$

Problem 3

a) Find a particular solution of the differential equation

$$x^2y'' - 4xy' + 4y = x^4, \quad x > 0.$$

b) Find a general solution of the differential equation

$$(y'' + 4y)'' = y'' + 4y.$$

Problem 4

The matrix $A = \begin{bmatrix} 1 & -1 & -2 & -2 & -4 \\ -2 & -4 & -2 & 1 & -1 \\ -2 & -1 & 1 & -2 & -1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$ and the vector $\mathbf{b} = \begin{bmatrix} -2 \\ 1 \\ -2 \\ 0 \end{bmatrix}$ are given.

a) Solve the system $A\mathbf{x} = \mathbf{b}$.

b) Find a basis for Null(A), Col(A) and Row(A).

Problem 5

Let the vectors $(1, 0, 1, -1)$, $(-1, 1, 0, 1)$, $(1, 1, 0, 1)$ be a basis for a subspace V of $\mathbb{R}^4$.

a) Use the Gram-Schmidt algorithm to find an orthogonal basis for V.

b) Find the orthogonal projection of $(1, 1, 3, 4)$ into V.

Problem 6

a) Show that the matrix $A = \begin{bmatrix} -1 & -1 & 1 \\ -1 & -3 & 1 \\ -1 & -3 & 1 \end{bmatrix}$ has eigenvalues $0, -1, -2$.

Find a matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

b) Find the solution $x_1(t)$, $x_2(t)$, $x_3(t)$ of the system of differential equations

$$\begin{aligned} x_1' &= -x_1 - x_2 + x_3 \\ x_2' &= -x_1 - 3x_2 + x_3 \\ x_3' &= -x_1 - 3x_2 + x_3 \end{aligned}$$

for which $x_1(0) = 1$, $x_2(0) = -1$, $x_3(0) = 2$.

Problem 7

Let A and B be $m \times n$ matrices, and let C be a matrix such that $A = BC$. Show that Col(A) is contained in Col(B). Also show that if C is invertible, then the rank of A equals the rank of B.

Norwegian University of Science and Technology
Department of Mathematics

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English version

TMA4115 Matematikk 3

7th june 2010
Time: 9:00
Examination Aids: D
No written and handwritten examination support materials are permitted.
Calculator: Citizen SR-270X or Hewlett Packard HP30S

Full reasoning should be given for the answers to all questions except question 5.

Problem 1.

a) Show that $|\mathrm{Re}\, z| \le |z|$.

b) Solve the equation $z^2 - 2\,\mathrm{i}\,z - 1 - 2\,\mathrm{i} = 0$. Write your answer in the form $x + \mathrm{i}\,y$.

Problem 2.

a) Solve the initial-value problem

$$y'' - 4y' + 3y = 0, \qquad y(0) = 1,\ y'(0) = 2.$$

b) Find a general solution of the differential equation

$$y'' - 4y' + 3y = 3 - 4e^x.$$

Problem 3.

Find a solution of the differential equation $y'' - (3x^2 + 4x^{-1})y' + (3x + 4x^{-2})y = 0$ which is linearly independent of the solution $y = x$.

Problem 4.

An underdamped spring (with mass 1) has equation of motion:

$$y'' + cy' + ky = 0$$

Two solutions of this differential equation are

$$y_1 = e^{\lambda t}\cos(\omega t), \qquad y_2 = e^{\lambda t}\sin(\omega t)$$

a) Compute the Wronskian $W(y_1, y_2)$ and find a formula which uses c and k instead of λ and ω.
Hint: Show that $\lambda = -c/2$ and $\omega^2 = k - c^2/4$.

b) Assume that the time between successive maxima is $2s$, and that the maximum amplitude is reduced to $1/4$ of its first value after 15 oscillations. Find the damping constant of the system.

Problem 5.

Multiple-choice question, answer without showing your reasoning with one alternative for each question.

Let A be a 4×3-matrix. What is Rank A? (Which alternative is always right?)

A: at most 3 B: 3 C: at least 3 D: 4

Which alternative is the least-squares solution $(\overline{x}, \overline{y})$ of the linear system

$$-x + y = 5, \ -x + 2y = 0, \ -3x + y = -5 \ ?$$

A: $(2, 3/2)$ B: $(1, 1)$ C: $(3/2, 3/2)$ D: $(2, 2)$

Problem 6.

Find a basis for each of the spaces Null(A), Col(A), Col(A)$^{\perp}$, and Row(A) for the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 1 \\ 3 & 6 & 1 & 0 & 2 & -1 \\ 4 & 8 & 2 & -2 & 0 & -4 \end{bmatrix}$$

Find the orthogonal projection of $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ on to Col(A).

Problem 7.

Let

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

a) Find the eigenvalues and eigenvectors of A.

b) Find a matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Can P be chosen such that $P^{-1} = P^T$?

c) Solve the system of differential equations

$$\begin{aligned} y_1' &= 3y_1' + y_2' + y_3' \\ y_2' &= y_1' + 2y_2' \\ y_3' &= y_1' + 2y_3' \end{aligned}$$

with initial position $y_1(0) = 3$, $y_2(0) = 2$, $y_3(0) = -2$.

Problem 8.

a) Let

$$A = \begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix}$$

Show that $A^2 = 0$ and that $I + A$ is invertible.

b) Let B be an $n \times n$-matrix such that $B^2 = 0$. Show that $I + B$ is invertible. Is B diagonalisable?