Norwegian University of Science and Technology
Department of Mathematical Sciences

Contact during the examination:
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EXAMINATION TMA4110 CALCULUS 3
English
Wednesday, 20 December 2011
Time: 9-13

Permitted aids (code C): Simple calculator (HP30S or Citizen SR-270X)
Rottman: *Matematisk formelsamling*

All answers should be justified: it should be made clear how the answer was obtained.

Problem 1 Solve the equation $z^2 + 4z + 4 + 2i = 0$. The answer should be given on the form $z = x + iy$.

Problem 2 A forced damped harmonic motion is described by the differential equation

$$y''(t) + 4y'(t) + 64y(t) = \cos \omega t.$$

a) Determine whether the motion is under-damped, is over-damped or is critically damped. Sketch (without computations) a solution of the homogeneous equation that satisfies the initial conditions $y(0) = 0, y'(0) = 1$.

b) Show that $y_p(t) = A \cos \omega t + B \sin \omega t$ is a particular solution of the equation when

$$A = \frac{64 - \omega^2}{(64 - \omega^2)^2 + 16\omega^2}, \quad B = \frac{4\omega}{(64 - \omega^2)^2 + 16\omega^2}.$$

c) Set $C = \max y_p(t)$. Which value of ω gives largest C? (You can use that $C = \sqrt{A^2 + B^2}$ without proving.)

Problem 3

a) Find the general solution of the equation

$$y'' + 2y' - 3y = 9t^2.$$

b) Find a particular solution of the equation

$$y'' + 2y' + y = \frac{e^{-t}}{t}, \ t > 0.$$

Problem 4 Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & -3 & 4 \\ 0 & 2 & 5 & -2 \end{bmatrix}.$$

a) Find a basis for the solution space Null(A) and a basis for the column space Col(A). Find the rank of A, rank(A).

b) For which values of a is $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ a \end{bmatrix}$ in Col(A)?

c) Let T be a linear transformation with standard matrix A. Mark each of the following statements true or false (the answers should be justified)

(1) T is a linear transformation from R^3 to R^4,

(2) T is a linear transformation from R^4 to R^3,

(3) T is onto,

(4) T is one-to-one.

Problem 5 Find a least-square solution of the system

$$\begin{array}{rrrcr} x & & +z & = & 0 \\ x & +2y & +3z & = & 5 \\ x & -2y & -z & = & 1 \\ & 4y & -z & = & -1 \end{array}$$

Problem 6 Let

$$A = \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}.$$

Show that $\mathbf{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$ is a (complex) eigenvector of A. Find the complex eigenvalues and complex eigenvectors of A.

Problem 7 Let A be an $n \times n$ matrix such that $A^2 = A$. Show that each vector $\mathbf{x}$ in R^n can be written on the form $\mathbf{x} = \mathbf{u} + \mathbf{v}$, where $A\mathbf{u} = \mathbf{u}$ and $A\mathbf{v} = 0$.

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EXAMINATION TMA4110 CALCULUS 3
English
Wednesday, 1 December 2010
Time: 9-13

Permitted aids (code C): Simple calculator (HP30S or Citizen SR-270X)
Rottman: *Matematisk formelsamling*

Results: 22 December 2010

All answers should be justified: it should be made clear how the answer was obtained. Each of the 12 problem parts (1, 2a, 2b, 3a, 3b, 4a, 4b, 4c, 5, 6a, 6b, 7) counts equally.

Problem 1 Write down the polar form of the complex number $w = \frac{3-i}{2i-1}$. Find all of the solutions of the equation $z^4 = w$ and draw the solutions on the complex plane.

Problem 2

a) The motion of a mechanical system is described by the differential equation

$$y'' + 6y' + 18y = 0.$$

Determine whether the motion is under-damped, is over-damped or is critically damped. Find a particular solution $y(t)$ that satisfies the initial conditions $y(0) = 0$, $y'(0) = 0.6$.

b) Find the steady-state solution of the equation

$$y'' + 6y' + 18y = 45\cos 3t.$$

Problem 3

a) Find the general solution of the equation

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = 0, \ x > 0.$$

b) Find a particular solution of the equation

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = x^2e^x, \ x > 0.$$

Problem 4 Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Show that A is invertible and find A^{-1}.

Problem 5 Let $V \subset \mathbf{R}^4$ be the solution space of the system of equations

$$\begin{aligned} x + y - z + w &= 0 \\ x + 2y - 2z + w &= 0 \end{aligned}$$

a) Find an orthogonal basis for V.

b) Find the orthogonal projection of $b = (1, 1, 1, 1)$ onto V.

c) Find an orthogonal basis for $\mathbf{R}^4$ where the first two vectors of this basis are the vectors you found in part a).

Problem 6 Let

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -t & 1 & 0 & 1 & 1 \\ 0 & -t & 1 & 0 & 1 \\ 0 & 0 & -t & -1 & 0 \end{bmatrix}.$$

a) Find the rank of M for each value of t.

b) For which values of t is there a 5×4 matrix L such that $ML = I$, where I is the identity matrix? (Remember to justify your answer).

Problem 7 The equation

$$3x^2 - 2xy + 3y^2 = 1$$

describes a conic section in xy-plane. Find a rotated coordinate system (x', y'), in which the equation of the conic section is of the form

$$\lambda_1(x')^2 + \lambda_2(y')^2 = 1.$$

What kind of conic section is it? Draw the new coordinate axis and the conic section in the xy-plane.

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EXAMINATION TMA4110 CALCULUS 3
English
Friday, 4 December 2009
Time: 9-13

Permitted aids (code C): Simple calculator (HP30S or Citizen SR-270X)
Rottman: *Matematisk formelsamling*

Results: 4 January 2010

All answers should be justified, it should be made clear how the answers were obtained. Each of the 12 problem parts (1, 2ab, 3ab, 4abc, 5, 6ab, 7) counts equal under grading.

Problem 1 Find all solutions of the equation

$$z^5 = \frac{16\left(2\sqrt{3} - 1 - i(2 + \sqrt{3})\right)}{2 - i},$$

and draw the solutions on the complex plane.

Problem 2

a) Solve the initial value problem

$$y'' - 6y' + 25y = 0, \quad y(0) = 1,\ y'(0) = -1.$$

b) Find general solution of the equation

$$y'' - 6y' + 25y = 20xe^x.$$

Problem 3

a) Find a particular solution of the equation

$$x^2y'' + xy' - y = 4x, \; x > 0.$$

b) The equation

$$x^2y'' - (2x + x^2)y' + (2 + x)y = 0, \; x > 0,$$

has a solution $y_1(x) = x$. Find another solution y_2 such that y_1 and y_2 are linearly independent. Calculate the Wronski determinant $W(y_1, y_2)$.

Problem 4 Let

$$A = \begin{bmatrix} 1 & -3 & 0 & 1 & 0 \\ -2 & 6 & -2 & 0 & -3 \\ 1 & -3 & 6 & -5 & 9 \end{bmatrix}.$$

a) Find a basis for null space Null(A) and a basis for the row space Row(A).

b) Find a basis for column space Col(A) and a basis for the orthogonal complement to Col(A), Col$(A)^{\perp}$.

c) Find the orthogonal projection of

$$\mathbf{b} = \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix}$$

into Col(A).

Problem 5 Let

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 0 \\ 1 & 2 & 3 & 4 \\ 5 & 0 & 0 & 6 \end{bmatrix}.$$

Find det(A) and solve the homogeneous system of equations $A\mathbf{x} = 0$. What is the rank of A?

Problem 6

a) Let

$$A = \begin{bmatrix} 7 & 24 \\ 24 & -7 \end{bmatrix}.$$

Find the eigenvalues of A and eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ such that the matrix P with column vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is an orthogonal matrix with determinant 1.

b) The equation

$$7x^2 + 48xy - 7y^2 - 40x - 30y = 0$$

describes a conic section in xy-plane. Find a rotated coordinate system (x', y'), in which the equation of the conic section is of the form

$$\lambda_1(x')^2 + \lambda_2(y')^2 + dx' + ey' = 0.$$

What kind of conic section is it? Draw the new coordinate axis and the conic section in the xy-plane.

Problem 7 Let A be a symmetric matrix and let $\mathbf{x}$ be an eigenvector of A. Show that if $\mathbf{y}$ is a vector that is orthogonal to $\mathbf{x}$ (it means $\mathbf{y} \cdot \mathbf{x} = 0$), then the vector $A\mathbf{y}$ is also orthogonal to $\mathbf{x}$.

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Examination in TMA4115 MATEMATIKK 3
English
Monday 6th June 2011
Time: 9-13

Examination Aids (code C): Calculator (HP30S or Citizen SR-270X)
Rottman: *Collection of Mathematical Formulas*

Sensur: 27th June 2011

Full reasoning should be given for all answers. Each of the 12 parts (1, 2ab, 3ab, 4ab, 5, 6ab, 7, 8) count equal weight to the final grade.

Problem 1 Find all complex solutions of the equation

$$z^3 = \frac{1+i}{1-i}.$$

Write the solutions in the form $re^{i\theta}$, and draw the solutions in the complex plane.

Problem 2

a) Find the solution of the homogeneous equation

$$y'' - y' - 2y = 0$$

with initial conditions $y(0) = 3$ and $y'(0) = 0$.

b) Find the general solution of the equation

$$y'' - y' - 2y = 8\sin x + 3e^{2x}.$$

Problem 3 Let $y_1(x) = \frac{1}{x}$ and $y_2(x) = x^{\frac{1}{2}}$ for $x > 0$.

a) Show that y_1 and y_2 are linearly independent for $x > 0$.

b) Find an Euler–Cauchy equation which has $y = c_1y_1 + c_2y_2$ as its general solution.

Problem 4 Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 3 & -1 \\ 2 & 5 & -2 & 2 & 7 & 0 \\ -2 & -3 & -2 & -2 & -2 & 10 \\ 1 & 1 & 2 & 1 & 4 & 1 \end{bmatrix}.$$

a) Find a basis for the null space, Null(A) and a basis for the row space, Row(A).

b) For which values of a does the following linear system have a solution? When it has a solution, how many does it have?

$$\begin{array}{rcrcrcrcrcrcr} x_1 & + & 2x_2 & & & + & x_4 & + & 3x_5 & - & x_6 & = & a \\ 2x_1 & + & 5x_2 & - & 2x_3 & + & 2x_4 & + & 7x_5 & & & = & 1 \\ -2x_1 & - & 3x_2 & - & 2x_3 & - & 2x_4 & - & 2x_5 & + & 10x_6 & = & -1 \\ x_1 & + & x_2 & + & 2x_3 & + & x_4 & + & 4x_5 & + & x_6 & = & 0 \end{array}$$

Problem 5 Let V be the column space of the matrix

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ -1 & -1 & -2 & 1 \end{bmatrix}$$

and let

$$\mathbf{b} = \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}.$$

Find the nearest point in V to $\mathbf{b}$ (that is, the orthogonal projection of $\mathbf{b}$ on to V).

Problem 6

a) Let

$$A = \begin{bmatrix} 8 & 3 \\ 2 & 7 \end{bmatrix}.$$

Find the eigenvalues and corresponding eigenvectors of A.

b) There are two places in Trondheim with bicycles that can be hired for free: Gløshaugen (G) and Torget (T). The bicycles can be hired from early in the morning and must be returned to one of the places the same evening. It is found that of the bicycles hired from G, 80% are returned to G and 20% to T. Of the bicycles hired from T, 30% are returned to G and 70% to T. We assume that this pattern is constant, that all bicycles are hired out each morning, and that no bicycles are stolen.

In the long term, what proportion of the bicycles will be at Gløshaugen each morning?

Problem 7 Find a (2×2)-matrix A such that the system of differential equations $\mathbf{x}' = A\mathbf{x}$ has general solution:

$$\mathbf{x}(t) = c_1 e^{-2t} \begin{bmatrix} 3 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -5 \\ 2 \end{bmatrix}.$$

Problem 8 Let A be an $(m \times n)$-matrix. Show that if $A\mathbf{x} = \mathbf{b}$ has solutions for all $\mathbf{b}$ in $\mathbb{R}^m$ then $A^T\mathbf{x} = \mathbf{0}$ has only the trivial solution.