(1) [Torsional creep problem.] Assume that $u \in C(\overline{\mathbb{D}}) \cap C^\infty(\overline{\mathbb{D}})$, $u|_{\partial \mathbb{D}} = 0$,

$$\Delta u = -2 \text{ in } \mathbb{D},$$

where $\mathbb{D} = \{(x,y) \mid x^2 + y^2 < 1\}$. Show that

$$\max_{\overline{\mathbb{D}}} |\nabla u| = \max_{\partial \mathbb{D}} |\nabla u|$$

<u>Hints:</u>

- $v = u(x,y) + \frac{1}{2}(x^2+y^2)$ is harmonic.
- $w = |\nabla u|^2$ is subharmonic.

(2) Let $\Omega \subset \mathbb{R}^2$ be a domain with finite area $|\Omega| = \iint_\Omega dx\,dy$. Show that

$$\frac{\iint_\Omega |\nabla \phi|^2 dx\,dy}{\iint_\Omega \phi^2 dx\,dy} \geq \frac{1}{|\Omega|}$$

for all $\phi \in C_0^\infty(\Omega)$, $\phi \not\equiv 0$. Conclude that the first eigenvalue λ_1 of Ω approaches ∞ if the area $|\Omega| \longrightarrow 0$.