① $\Delta\left[u(x_1,t)u(x_2,t)\cdots u(x_n,t)\right]$

$$= \frac{\partial^2 u(x_1,t)}{\partial x_1^2}\cdot u(x_2,t)\cdots u(x_n,t)$$

$$+ u(x_1,t)\frac{\partial^2 u(x_2,t)}{\partial x_2^2}\cdots u(x_n,t) \quad \cdots\cdots$$

$$+ u(x_1,t)\,u(x_2,t)\cdots\frac{\partial^2 u(x_n,t)}{\partial x_n^2}$$

$$= \frac{\partial u(x_1,t)}{\partial t}u(x_2,t)\cdots u(x_n,t) + u(x_1,t)\frac{\partial u(x_2,t)}{\partial t}\cdots u(x_n,t)$$

$$+ \cdots + u(x_1,t)u(x_2,t)\cdots\frac{\partial u(x_n,t)}{\partial t}$$

$$= \frac{\partial}{\partial t}\left(u(x_1,t)u(x_2,t)\cdots u(x_n,t)\right) \quad \text{i.e.} \quad \boxed{\frac{\partial u}{\partial t} = \Delta u}$$

② By KIRCHHOFF's formula

$$u(10,0,0,t) = \frac{1}{4\pi t}\iint_{\partial B(10,0,0;t)} h\, dS_t$$

where $h = 5$ in $\overline{B(0,0,0;3)}$ and $h = 0$ otherwise. The sphere $\partial B(10,0,0;t)$ intersects the ball $\iff 7 \le t \le 13$. It follows that

$$u(10,0,0,t) \neq 0 \iff 7 < t < 13.$$

③ Integrating the usual mean value formula

$$4\pi r^2 u(\vec{x}_0) = \iint_{\partial B(\vec{x}_0,r)} u\, dS_r \qquad (*)$$

over the interval $[0,R]$ we get

$$\frac{4}{3}\pi R^3 u(\vec{x}_0) = \iiint_{B(\vec{x}_0,R)} u(\vec{y})\, d^3\vec{y} \qquad (\dagger)$$

provided that $\overline{B(\vec{x}_0, R)} \subset \Omega$. — We can also differentiate (†) with respect to R to arrive at (*), now with $r = R$. (See page 164 in $\mathbb{B}$.)

9.1 $\quad \Delta u \geq 0, \quad \Delta \phi = 0, \quad u|_{\partial\Omega} = \phi|_{\partial\Omega}$

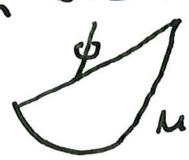
Now $\quad \Delta(u - \phi) = \Delta u - \Delta \phi \geq 0 - 0 = 0$

Hence $u - \phi$ is subharmonic and hence

$$\max_{\bar{\Omega}} (u - \phi) = \max_{\partial\Omega} (u - \phi) = 0$$

by the Maximum Principle (Theorem 9.5). Thus

$$u \leq \phi \text{ in } \Omega.$$



9.3 $\quad \Omega \subset B(\bar{0}, R),$

$$\Delta u = -f \text{ in } \Omega, \quad u|_{\partial\Omega} = 0.$$

Here $f \in C(\bar{\Omega})$, $\quad u \in C^2(\Omega), \quad u \in C(\bar{\Omega})$.

$$\Delta[u(\bar{x}) + c|\bar{x}|^2] = \Delta u(\bar{x}) + 2nc$$

$$= -f(\bar{x}) + 2nc \geq -\operatorname{Max}_{\bar{\Omega}}(f) + 2nc$$

$$\geq 0, \quad \text{if} \quad c = \frac{1}{2n} \operatorname{Max}_{\bar{\Omega}}(f)$$

For this value of c (and all greater values) $u(\bar{x}) + c|\bar{x}|^2$ is subharmonic and so it obeys the maximum principle

$$\max_{\bar{\Omega}} (u(\bar{x}) + c|\bar{x}|^2) = \max_{\partial\Omega} (u(\bar{x}) + c|\bar{x}|^2)$$

$$\leq \max_{\partial\Omega}(u) + \max_{x\in\partial\Omega}(c|\bar{x}|^2)$$

$$\leq 0 + cR^2 = \frac{1}{2n} R^2 \, \mathrm{Max}(f)$$

Assume $c \geq 0$. Otherwise $\Delta u \geq 0$ and $u \leq 0$.

It follows that

$$u \leq \frac{R^2}{2n} \mathrm{Max}(f).$$

9.4 Harnack's Inequality

$u \geq 0$, $\Delta u = 0$ in a ball $B(0, 4R+\varepsilon)$

Let $\bar{x}, \bar{y} \in B(\bar{0}, R)$. By the solid mean value property

$$\mathrm{vol}(B_{3R})\, u(\bar{y}) = \int_{B(\bar{y},3R)} u \, d^3\bar{z} \geq \int_{B(\bar{x},R)} u \, d^3\bar{z}$$

$$= \mathrm{vol}(B_R)\, u(\bar{x}). \implies$$

$$u(\bar{x}) \leq 3^n u(\bar{y})$$

Since the points were arbitrary

$$\max_{B(0,R)} u \leq 3^n \min_{B(0,R)} u$$

Independent of u!

(In particular, $u \geq 0$ cannot have any zeros, except for $u \equiv 0$.)