(6.1)

$$\begin{cases} \dfrac{\partial u}{\partial t} - \gamma u - \Delta u = 0 \\ u(\bar{x}, 0) = f(\bar{x}) \quad \text{in } \mathbb{R}^n \end{cases}$$

(α and γ are constants)

$u(\bar{x}, t) = e^{\alpha t} v(\bar{x}, t) \qquad u(\bar{x}, 0) = f(\bar{x}) = v(\bar{x}, 0)$

The eqn becomes

$$e^{\alpha t}\{v_t + \cancel{\alpha v} - \cancel{\gamma v} - \Delta v\} = 0$$

The favourable choice $\alpha = \gamma$ yields

$$v_t = \Delta v$$

and so

$$u(x, t) = \frac{e^{\gamma t}}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} f(\bar{y}) e^{-\frac{|\bar{x} - \bar{y}|^2}{4t}} d^n \bar{y}$$

(6.3)

$$\frac{\partial u}{\partial t} = \Delta u \quad \text{in } \Omega \times (0, \infty)$$

$u \in C^2(\Omega \times (0, \infty)), \quad u \in C^1(\bar{\Omega} \times (0, \infty))$

a) $\dfrac{\partial u}{\partial n} = 0$ on $\partial\Omega \times (0, \infty)$, $\bar{n}$ = outer normal

$$\mathcal{U}(t) = \int_\Omega u(\bar{x}, t)\, d^n \bar{x}$$

$$\mathcal{U}'(t) = \int_\Omega u_t(\bar{x}, t)\, d^n \bar{x} = \int_\Omega \Delta u \, d^n \bar{x}$$

$$= \int_{\Omega} \operatorname{div}(\nabla u)\, d^n\bar{x} = \oint_{\partial\Omega} \frac{\partial u}{\partial n}\, dS = \oint_{\partial\Omega} 0\, dS$$

$= 0$. Hence $\mathcal{U}(t) =$ Constant.

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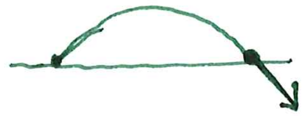
b) $\begin{cases} u > 0 & \text{in } \Omega \times (0,\infty) \\ u = 0 & \text{on } \partial\Omega \times (0,\infty) \end{cases}$



It follows that $\frac{\partial u}{\partial n} \leq 0$

Hence $\mathcal{U}'(t) = \oint_{\partial\Omega} \frac{\partial u}{\partial n}\, dS \leq 0$

and thus $\mathcal{U}(t)$ is decreasing (as t increases).

6.4 $\quad \frac{\partial u}{\partial t} = \Delta u$ in $\mathbb{R}^n \times (0,\infty)$ -

Assume that $u \in C^2(\Omega \times (0,\infty))$ and that u and ∇u are continuous in $\bar{\Omega} \times (0,\infty)$.

a) $u\big|_{\partial\Omega} = 0$. $\quad \eta(t) = \int_{\Omega} u(\bar{x},t)^2\, d^n\bar{x}$

$$\eta'(t) = 2\int_{\Omega} u\, u_t\, d^n\bar{x}$$

$$= 2\int u\, \Delta u\, d^n\bar{x} = -2\int_{\Omega} |\nabla u|^2 d^n x + 2\oint_{\partial\Omega} u \frac{\partial u}{\partial n}\, dS$$

$$\operatorname{div}(u\nabla u) = |\nabla u|^2 + u\Delta u$$

$$\eta'(t) = -2\int_\Omega |\nabla u|^2 d^n\bar{x} \quad \text{if } u|_{\partial\Omega} = 0$$

$$\leq 0 \implies \eta(t) \text{ is decreasing.}$$

b) If there are two such solutions, say u_1 and u_2, then $u = u_2 - u_1$ solves the problem

$$\begin{cases} u_t = \Delta u \\ u|_{\partial\Omega} = 0;\ t \geq 0 \\ u(\bar{x},0) = 0, \quad x \in \Omega \end{cases}$$ (assume continuity at $t = 0$)

Then, by case a),

$$\int_\Omega u(\bar{x},t)^2 d^n x \leq \int_\Omega u(\bar{x},0)^2 d^n x = 0$$

and so $u(\bar{x},t) = 0$. Thus $u_1 = u_2$.