Problem 1

Suppose that $u_1(s,t),\ u_2(s,t), \ldots,\ u_n(s,t)$ are solutions of the one-dimensional Heat Equation. Show that the product

$$u(\overline{x},t) = u(x_1, x_2, \ldots, x_n, t) = u(x_1,t)\cdots u(x_n,t)$$

solves the Heat Equation $u_t = \Delta u$. As usual, $x = (x_1, x_2, \ldots, x_n)$ is a point in $\mathbf{R}^n$. A good example is

$$\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|\overline{x}|^2}{4t}} = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x_1^2}{4t}} \cdots \frac{1}{\sqrt{4\pi t}} e^{-\frac{x_n^2}{4t}}.$$

Problem 2

Let $u = u(x,y,z,t)$ be the solution to the problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \qquad c = 1 \\ u(x,y,z,0) &= 0 \\ \frac{\partial u}{\partial t}\Big|_{t=0} &= \begin{cases} 5, & \text{when } x^2+y^2+z^2 \le 9, \\ 0, & \text{when } x^2+y^2+z^2 > 9 \end{cases} \end{aligned}$$

When is $u(10,0,0,t) \neq 0,\ t > 0$? (Huygens' Principle.)

Problem 3 (Solid Mean Value Property). Prove that $u \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^3$, satisfies $\Delta u = 0$ in Ω $\iff$

$$u(\overline{x}) = \frac{1}{\frac{4}{3}\pi r^3} \iiint_{B(\overline{x},r)} u(\overline{y})\, d^3\overline{y} \quad \text{whenever } \overline{B}(x,r) \subset \Omega$$

B § 9.6: 9.1, 9.3, 9.4.