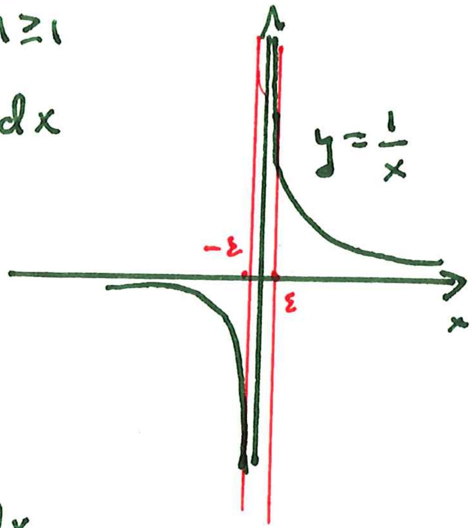
(12.1) $\langle u, \varphi \rangle = \int_{-1}^{1} \frac{\varphi(x)-\varphi(0)}{x}\,dx + \int_{|x|\geq 1} \frac{\varphi(x)}{x}\,dx$



$= \lim_{\varepsilon \to 0} \int_{\varepsilon<|x|<1} \frac{\varphi(x)-\varphi(0)}{x}\,dx + \int_{|x|\geq 1} \frac{\varphi(x)}{x}\,dx$

$= \lim_{\varepsilon \to 0} \int_{\varepsilon<|x|<1} \frac{\varphi(x)}{x}\,dx + \int_{|x|\geq 1} \frac{\varphi(x)}{x}\,dx$

$= \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\,dx = PV \int_{-\infty}^{\infty} \frac{\varphi(x)}{x}\,dx$

(2) a) not a distribution, because it is not linear.

b) ——— " ———.

d) $T(\phi) = \int_{-\infty}^{\infty} |x|^2 \varphi(x)$ defines a distribution. First,

$|T(\varphi)| \leq \max|\varphi(x)| \int_a^b |x|^2 dx$ if $\text{supp}\,\varphi \subset [a,b]$. It is obviously <u>linear</u>.

<u>Continuity</u>: $\varphi_k \xrightarrow{\mathcal{D}} \varphi \implies$ There is some interval containing all the supports, say $\text{supp}(\varphi_k) \subset [a,b]$, and $\max|\varphi_k - \varphi| \longrightarrow 0$.

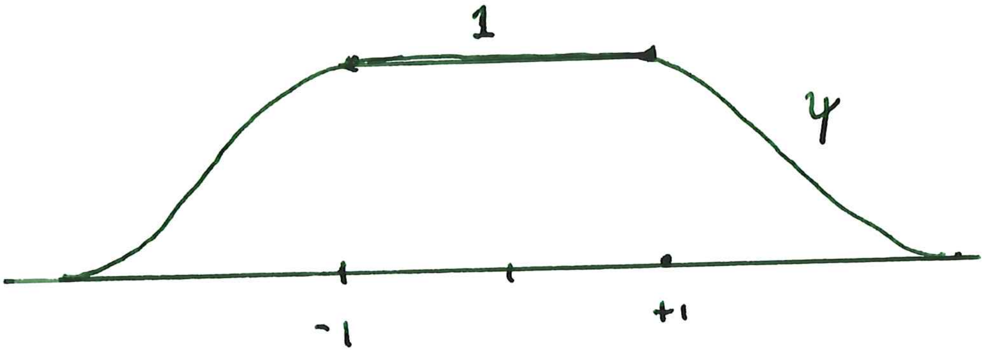
Thus

$|T(\varphi_k) - T(\varphi)| \leq \int_a^b |x|^2 [\varphi_k(x) - \varphi(x)]\,dx$

$\leq \|\varphi_k - \varphi\|_{L^\infty} \int_a^b x^2 dx \longrightarrow 0$ as $k \longrightarrow \infty$.

2c) $$T(\varphi) = \sum_{n=0}^{\infty} \varphi^{(n)}(0)$$

is not a distribution, because the series diverges for some $\varphi \in \mathcal{D}(\mathbb{R})$. To see this, assume that $\psi \in C_0^\infty([-2, +2])$ is identically $= 1$ in $[-1, 1]$



Then $\varphi(x) = e^x \psi(x)$ is an admissible test-function, and

$$\frac{d^n \varphi(x)}{dx^n} = e^x, \quad \text{when } |x| < 1.$$

It follows that $T(\varphi) = 1 + 1 + 1 + \cdots = \infty$.

Remark: ψ can be constructed as a suitable convolution $\rho_\varepsilon * H$, where $H(x) = 1$, when $|x| \le 1{,}5$ and $H|x| = 0$ when $|x| \geq 2$. We omit this construction here.

H

(12.2)

$$f(x) = \begin{cases} \log(x), & x > 0 \\ -\log(-x), & x < 0 \end{cases} \quad ; \quad f'(x) = \frac{1}{|x|}, \quad x \neq 0$$

$$\langle f, \varphi' \rangle = + \int f(x)\, \varphi'(x)\, dx$$

$$= \left(\int_{-\infty}^{-1} + \int_{-1}^{0} + \int_{0}^{1} + \int_{1}^{\infty} \right) f(x)\, \varphi'(x)\, dx$$

$$\int_{-\infty}^{-1} f(x)\varphi'(x)\, dx = \underbrace{\Big/_{-\infty}^{-1} f(x)\varphi(x)}_{=0} - \int_{-\infty}^{-1} \frac{\varphi(x)}{|x|}\, dx$$

$$\int_{1}^{\infty} f(x)\, \varphi'(x)\, dx = 0 - \int_{1}^{\infty} \frac{\varphi(x)}{|x|}\, dx$$

$$\int_{-1}^{-\varepsilon} f(x)\, \varphi'(x)\, dx = \int_{-1}^{-\varepsilon} f(x) \frac{d}{dx}\left(\varphi(x) - \varphi(0)\right) dx$$

$$= \Big/_{-1}^{-\varepsilon} f(x)\left(\varphi(x) - \varphi(0)\right) - \int_{-1}^{-\varepsilon} \frac{\varphi(x) - \varphi(0)}{|x|}\, dx$$

$$= -\log(\varepsilon)\big(\underbrace{\varphi(\varepsilon) - \varphi(0)}_{\sim \varepsilon\varphi'(0)}\big) - \int_{-1}^{-\varepsilon} \frac{\varphi(x) - \varphi(0)}{|x|}\, dx$$

$$\longrightarrow - \int_{-1}^{0} \frac{\varphi(x) - \varphi(0)}{|x|}\, dx$$

$$\int_{+\varepsilon}^{1} f(x)\varphi'(x)\, dx \xrightarrow[\varepsilon \to 0+]{} - \int_{0}^{1} \frac{\varphi(x) - \varphi(0)}{|x|}\, dx$$

$$\left\langle \frac{df}{dx}, \varphi \right\rangle = \int_{-1}^{1} \frac{\varphi(x) - \varphi(0)}{|x|}\, dx + \int_{|x| \geq 1} \frac{\varphi(x)}{|x|}\, dx$$

↑ The derivative of the distribution f.