(4.5) Telegraph equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + a \frac{\partial u}{\partial t} + b u = 0$$

Let

$$u(x,t) = e^{-\frac{at}{2}} w(x,t).$$

A pure calculation yields

$$\frac{\partial^2 w}{\partial t^2} - c^2 \frac{\partial^2 w}{\partial x^2} + \left[b - \frac{a^2}{2} \right] w = 0.$$

(This is the Klein-Gordon eqn..) This reduces to the wave eqn, if

$$b = \frac{a^2}{2}.$$

(4.7)

a) $$\frac{\partial u}{\partial t} - i \Delta u = 0$$ SCHRÖDINGER

$$\frac{d}{dt} \int_{B_R} |u(x,t)|^2 d^n \bar{x} = 2 \int_{B_R} (u_t \bar{u} + u \bar{u}_t)\, d^n \bar{x}$$

$$= 2 \int_{B_R} (+i \bar{u} \Delta u - i u \overline{\Delta u})\, d^n \bar{x}$$ [Divergence $\Rightarrow$ thm]

$$= 2i \oint_{\partial B_R} \underbrace{(\bar{u} \nabla u - u \overline{\nabla u})}_{2i\, \mathrm{Im}(\bar{u} \nabla u)} \cdot \vec{n}\, dS_R = -4 \oint_{\partial B_R} \mathrm{Im}(\bar{u} \nabla u) \cdot \vec{n}\, dS_R$$

$\bar{n}$ = normal

$$= -4 \oint_{\partial B_R} \nabla(|u|^2) \cdot \vec{n}\, dS_R$$

Assuming the decay $$\lim_{R \to \infty} \oint_{\partial B_R} \nabla(|u|^2) \cdot \bar{n}\, dS = 0$$

we conclude that $$\frac{d}{dt} \int_{\mathbb{R}^n} |u(\bar{x},t)|^2 d^n \bar{x} = 0.$$

Thus "the energy"

$$\int_{\mathbb{R}^n} |u(\bar{x},t)|^2 d^n\bar{x} = \text{Constant}.$$

and

$$\int_{\mathbb{R}^n} |u(\bar{x},t)|^2 d^n\bar{x} = \int_{\mathbb{R}^n} |g(\bar{x})|^2 d^n\bar{x}.$$

b) If we have two solutions with initial values $u_1(\bar{x},t) = g(\bar{x})$, $u_2(\bar{x},t) = g(\bar{x})$, then the solution $u = u_2 - u_1$ has initial values zero. By case a)

$$\int_{\mathbb{R}^n} |u_2(\bar{x},t) - u_1(\bar{x},t)|^2 d^n\bar{x} = 0$$

and so $u_2 = u_1$. Also here the decay assumption seems to be needed.

(4.3) $\frac{\partial^2 u}{\partial t^2} - \Delta u + m^2 u = 0$ Klein-Gordon

a) **Ansatz**: $u(\bar{x},t) = e^{i(\bar{k}\cdot\bar{x} - \omega t)}$, ω = real number

We get

$$(\omega^2 - |\bar{k}|^2 + m^2)\, e^{i(\bar{k}\cdot\bar{x} - \omega t)} = 0,$$

$$\omega^2 = |\bar{k}|^2 - m^2, \qquad |\bar{k}| \geq m$$

Remark: If t is fixed, such a solution is constant on the plane $\bar{k}\cdot\bar{x} = \omega t$. The vector $\bar{k}$ is the normal of the plane.

"plane waves"

4.9.6

$$\mathcal{E}(t) = \frac{1}{2}\int \left(u_t(x,t)^2 + |\nabla u(\bar{x},t)|^2 + m^2 u(\bar{x},t)^2 \right) d^n\bar{x}$$

$$\dot{\mathcal{E}}(t) = \int \left(u_t u_{tt} + \nabla u \cdot \nabla u_t + m^2 u u_t \right) d^n\bar{X}$$

$$= \int u_t \big(\underbrace{u_{tt} - \Delta u + m^2 u}_{=0} \big) d^n\bar{x} + \oint (u_t \nabla u \cdot \bar{n})\, dS$$

$= 0$ provided that $\oint (u_t \nabla u \cdot \bar{n})\, dS$ decays at infinity ($|\bar{x}| \to \infty$).

EXTRA.

$$u_{tt} - \overset{\substack{\text{DIFFUSION}\\ \downarrow}}{u_{xx}} + \underset{\substack{\uparrow\\ \text{DISSIPATION}\\ \text{for } \alpha>0}}{\alpha u_t} + \overset{\substack{\text{convection}\\ \text{(drift term)}\\ \downarrow}}{\beta u_x} + \underset{\substack{\uparrow\\ \text{DISPERSION}}}{\gamma u} = 0$$

$$\begin{cases} u_{tt} - c^2 \Delta u = F(\bar{x},t) & \text{in } \mathbb{R}^3 \times (0,\infty) \\ u(\bar{x},0) = 0, & \bar{x} \in \mathbb{R}^3 \\ u_t(\bar{x},0) = 0, & \bar{x} \in \mathbb{R}^3 \end{cases}$$

From Duhamel's principle we get

SOLUTION $$u(\bar{x},t) = \frac{1}{4\pi c^2} \iiint_{B(\bar{x},ct)} \frac{F\left(\bar{y}, t - \frac{|x-y|}{c}\right)}{|\bar{x}-\bar{y}|}\, d^3\bar{y}$$

"RETARDED POTENTIAL"