(1) $$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = -2 & \text{in } \Omega \\ u|\partial\Omega = 0 \end{cases} \qquad u \in C^1(\overline{\Omega}) \cap C^3(\Omega)$$

If Ω is the unit disc $x^2+y^2<1$, then the result follows from the fact that

$$u(x,y) = \frac{1}{2} - \frac{x^2+y^2}{2}$$

PS It was a mistake in the text to use $\mathbb{D}$ as Ω.

is the unique solution (easy to prove). — Below a proof valid for an arbitrary bounded domain Ω.

- $v(x,y) = u(x,y) + \frac{1}{2}(x^2+y^2)$ is harmonic. All derivatives of harmonic functions are themselves harmonic. Thus $v_x - x$ and $v_y - y$ are also harmonic:

$$\Delta(v_x - x) = 0, \quad \Delta(v_y - y) = 0$$

Remark
$u|\partial\Omega = 0$ not needed!

- $|\nabla u|^2 = (v_x - x)^2 + (v_y - y)^2$

- $\Delta(fg) = f\Delta g + 2\nabla f \cdot \nabla g + g \Delta f$

$$\begin{aligned} \Delta(|\nabla u|^2) &= 2\nabla(v_x - x)\cdot\nabla(v_x - x) \\ &\quad + 2\nabla(v_y - y)\cdot\nabla(v_y - y) \\ &= 2|\nabla(v_x - x)|^2 + 2|\nabla(v_y - y)|^2 \geq 0 \end{aligned}$$

Hence $|\nabla u|^2$ is subharmonic and by the maximum principle

$$\max_{\overline{\Omega}} |\nabla u|^{\cancel{2}} \leq \max_{\partial\Omega} |\nabla u|^{\cancel{2}}. \qquad \square$$

(2) The result follows from

$$\iint_\Omega |\varphi|^2 dx\,dy \overset{*)}{\leq} \left(\iint_\Omega |\nabla\varphi|\,dx\,dy\right)^2$$

$$\underset{\text{HÖLDER}}{\leq} |\Omega| \iint_\Omega |\nabla\varphi|^2 dx\,dy$$

$|\nabla\phi|^2 = \varphi_x^2 + \varphi_y^2$

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*) $$\varphi(x,y) = \int_{-\infty}^{x} \varphi_x(t_1, y)\,dt_1$$

$$|\varphi(x,y)| \leq \int_{-\infty}^{\infty} |\varphi_x(t_1, y)|\,dt_1 \leq \int_{-\infty}^{\infty} |\nabla\varphi(t_1,y)|\,dt_1$$

Also

$$|\varphi(x,y)| \leq \int_{-\infty}^{\infty} |\nabla\varphi(x, t_2)|\,dt_2$$

Multiply and "integrate away" x and y:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\varphi(x,y)|^2 dx\,dy \leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\nabla\varphi(t_1,y)|\,dt_1\,dy \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\nabla\varphi(x,t_2)|\,dx\,dt_2$$

$$= \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\nabla\varphi(x,y)|\,dx\,dy\right)^2. \quad \square$$

Finally

$$\lambda_1 = \inf_\varphi \frac{\iint |\nabla\phi|^2}{\iint |\varphi|^2} \geq \frac{1}{|\Omega|}$$

(Small domains have large eigenvalues!)

(11.5) $\Delta \phi + \lambda_1 \phi = 0$ in Ω

$\phi \in C(\Omega) \cap H_0^1(\Omega)$

$\phi = \phi_+ - \phi_-$, $\phi_+ \geq 0$, $\phi_- \geq 0$, $\phi_+ \phi_- = 0$

We assume that we know $\phi_+ \in H_0^1(\Omega)$,

$$\nabla \phi_+ = \begin{cases} \nabla \phi, & \text{if } \phi > 0 \\ 0 & \text{a.e. where } \phi_+ = 0 \end{cases}$$

The same for ϕ_-. It follows that

$\nabla \phi_+ \cdot \nabla \phi_- = 0$ a.e..

(*)
$$\lambda_1 = \frac{\int |\nabla \phi|^2}{\int \phi^2} = \frac{\int |\nabla \phi_+|^2 + \int |\nabla \phi_-|^2 - 2\int \nabla \phi_+ \cdot \nabla \phi_-}{\int \phi_+^2 + \int \phi_-^2 - 2\int \phi_+ \phi_-}$$

$$= \frac{\int |\nabla \phi_+|^2 + \int |\nabla \phi_-|^2}{\int \phi_+^2 + \int \phi_-^2}$$

$$\lambda_1 \leq \frac{\int |\nabla \phi_+|^2}{\int \phi_+^2}, \quad \lambda_1 \leq \frac{\int |\nabla \phi_-|^2}{\int \phi_-^2}$$

<u>Claim:</u> $\phi_+ \equiv 0$ or $\phi_- \equiv 0$.

If not, proceed as to the left.

For positive numbers

$$\min\left\{\frac{A}{a}, \frac{B}{b}\right\} \leq \frac{A+B}{a+b} \leq \max\left\{\frac{A}{a}, \frac{B}{b}\right\}$$

If the smaller one is with $+$-sign, we have

$$\frac{\int |\nabla \phi_+|^2}{\int \phi_+^2} \leq \frac{\int |\nabla \phi_+|^2 + \int |\nabla \phi_-|^2}{\int \phi_+^2 + \int \phi_-^2} \leq \frac{\int |\nabla \phi_+|^2}{\int \phi_+^2}$$

But this implies that

$$\lambda_1 = \frac{\int |\nabla \phi_+|^2}{\int \phi_+^2}$$

Then it follows that also $\lambda_1 = \dfrac{\int |\nabla \phi_-|^2}{\int \phi_-^2}$.

The same, if the smaller ratio comes for ϕ_-. Since ϕ_+ and ϕ_- are minimizing they are first eigenfunctions:

$$\Delta \phi_\pm + \lambda_1 \phi_\pm = 0 \quad \text{in } \Omega.$$

They are both <u>superharmonic</u>. The <u>strong</u> minimum principle for $\phi_+ \geq 0$ shows that $\phi_+ > 0$, if $\phi_+ \not\equiv 0$. Then $\phi = \phi_+$. (For ϕ_- we get that $\phi = -\phi_-$ or $\phi_- \equiv 0$.)

We have shown that an arbitrary <u>first</u> eigenfunction is zero-free.

Let u be an arbitrary first eigenfunct. So is $u(x) + c\phi(x)$. Let $x_0 \in \Omega$. Then the first eigenfunction

$$u(x) - \frac{u(x_0)}{\phi(x_0)}\phi(x)$$

has a zero, namely x_0. This contradiction can be avoided only if

$$u(x) \equiv C\phi(x) \quad \square$$

Hence the family $C\phi(x)$ contains all first eigenfunctions, i.e. λ_1 is <u>simple</u>.