NTNU
Norwegian University of Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4295 Statistical inference**

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Examination date: November 28, 2023

Examination time (from–to): 15:00 – 19:00

Permitted examination support material: C

- Tabeller og formler i statistikk, Akademika
- Mathematische Formelsamlung (Matematisk formelsamling) by K. Rottmann
- Stamped yellow A5 sheet with your own handwritten notes
- A specific basic calculator.

Other information:

You may write in English or Norwegian.

All answers must be justified. The answers must include enough details to see how they have been obtained. You must, as always, formulate necessary assumptions as part of the proof of a claim.

All 10 sub-problems carry the same weight for grading.

Language: English

Number of pages: 2

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
sort/hvit ☐	**farger** ☒
skal ha flervalgskjema ☐	

Date Signature

Problem 1 Let R be a random variable with a density $f(r) = re^{-\frac{r^2}{2}}$ for $r > 0$. The Rayleigh* distribution is the distribution of the random quantity σR where R is as above and $\sigma > 0$. Let the data $r_1, \ldots, r_n$ be a random sample from the Rayleigh distribution where σ is an unknown model parameter. An experiment † with $n = 10$ gives the estimate $\hat{\sigma} = \sqrt{(r_1^2 + \cdots + r_n^2)/(2n)} = 4.5\,\mathrm{m}$.

Figure 1: The SI brochure states that the uncertainty should be stated according to the ISO Guide to the expression of Uncertainty in Measurements (GUM).

a) Verify that f is a probability density. Prove that R^2 has a gamma distribution.

b) Find the maximum likelihood estimate of σ.

c) Show how $\hat{\sigma}$ can be used to calculate the p-value p for a test of the hypothesis $H_0\colon \sigma \geq \sigma_0$ where $\sigma_0 = 5\,\mathrm{m}$. An app gives $p = 0.296$. What is the conclusion of the test with level $\alpha = 5\,\%$? Is this the uniformly most powerful test?

d) Find the uniform minimum variance unbiased estimator of σ. Determine the standard uncertainty of the corresponding estimate and compare with the Cramér–Rao lower bound. Note: Numerical answers are not required here!

*John William Strutt, 3rd Baron Rayleigh, (1842–1919) was a British mathematician and physicist. He received the 1904 Nobel Prize in Physics. His textbook *The Theory of Sound* (1877) is still used today as a standard reference by acousticians and engineers.

†The experiment can be used to determine if a GPS using the new dual frequency mode has an uncertainty below $\sigma_0 = 5\,\mathrm{m}$.

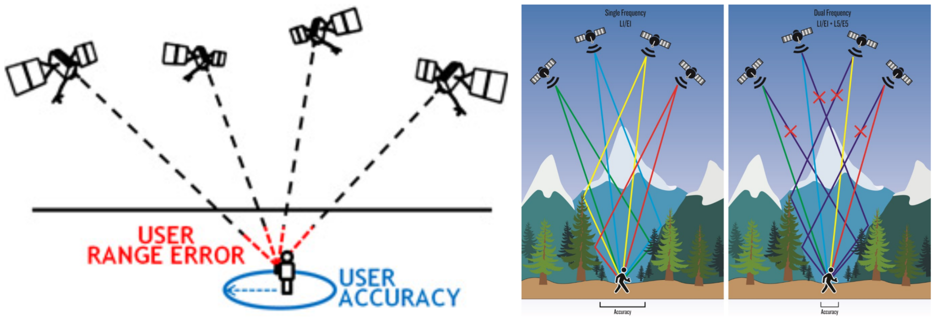

Figure 2: The position of a rover can be found by distance measurements.

Problem 2 A GPS measurement gives the position of the entrance of the NTNU examination house at Sluppen to be $x = 569\,701\,\mathrm{m}$ easting and $y = 7\,030\,503\,\mathrm{m}$ northing in the coordinate system defined by Universal Transverse Mercator grid zone 32N (EU89, UTM32). Assume the coordinate estimators X and Y to be unbiased and independent with a normal distribution and a common unknown standard deviation σ. Let (μ_X, μ_Y) be the unknown true coordinates of the entrance. The data is the observed coordinates (x, y) together with the data $(r_1, \ldots, r_n)$ from **Problem 1** with the same σ.

a) Let $\hat{\sigma}$ be as in **Problem 1**. Show that $(x, y, \hat{\sigma})$ is a minimal sufficient statistic. Is this the maximum likelihood estimate?

b) Let $g(x, y, r_1, \ldots, r_n) = (a + cx, b + cy, cr_1, \ldots, cr_n)$. Give requirements on the quantities a, b, c such that this defines a group $\mathcal{G}$ of transformations. What is the identity group element? Find formulas for g^{-1} and $g_1 \circ g_2$.

c) Show that the family of probability densities for the data is invariant under $\mathcal{G}$. How can this be useful?

d) Prove that $(x, y, \hat{\sigma})$ is a complete sufficient statistic. How can this be useful?

Assume that the standard deviation of the coordinate estimators is known and equals $\sigma = 5\,\mathrm{m}$ for the remainder of this problem.

e) Prove that $Z = (X - \mu_X, Y - \mu_Y)$ is a pivot. Use this to determine a 95 % confidence region for (μ_X, μ_Y) based on the GPS measurement. Give an interpretation of the result.

f) Can the hypothesis $H_0\colon \mu_X = 569\,705\,\mathrm{m}, \mu_Y = 7\,030\,500\,\mathrm{m}$ be rejected using the likelihood test with a 5 % level?