**Problem 1.** (10 points)

For each of the following statements, decide whether it is true or false. If a statement is false, provide a counterexample (you need not provide proofs of true statements).

a) Every Cauchy sequence in a complete metric space converges.

b) Assume that $U$ is a Banach space and that $F\colon U \to U$ is such that $\|F(u) - F(v)\| \leq \|u - v\|$ for all $u, v \in U$. Then $F$ has a fixed point in $U$.

c) The range of a linear transformation $T\colon U \to V$ between vector spaces $U$ and $V$ is a subspace of $V$.

d) Let $U$ be a vector space and let $T\colon U \to U$ be bounded linear and surjective. Then $T$ is invertible.

**Problem 2.** (10 points)

Define the following notions:

a) Assume that $(U, \|\cdot\|_U)$ is a normed space. Define the notion of an *open set* in $U$.

b) Let $(U, \|\cdot\|_U)$ and $(V, \|\cdot\|_V)$ be normed spaces. Define the notion of a *bounded linear mapping* from $U$ to $V$.

c) Let $X$ be a non-empty set. Define the notion of a metric on $X$.

d) Let $(X, d)$ be a metric space. Define the notion of a Cauchy sequence in $X$.

**Problem 3.** (10 points)

Define

$$A = \begin{pmatrix} 3 & 0 & 6 & -6 \\ 0 & -2 & 4 & 4 \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} 9 \\ 6 \end{pmatrix}.$$

a) Find the singular value decomposition of the matrix $A$.

b) Find $x \in \mathbb{R}^4$ with $Ax = b$ such that $\|x\|_2$ is minimal.

**Problem 4.** (10 points)

Assume that $T\colon \mathbb{C}^n \to \mathbb{C}^n$ is a linear transformation with minimal polynomial

$$p(z) = (z-2)z(z+2)(z+5).$$

Find the minimal polynomial of $T^2$.

**Problem 5.** (10 points)

Assume that $F\colon \mathbb{R} \to \mathbb{R}$ is a contraction and that $g\colon \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$\int_{-\infty}^{\infty} |g(x)|\, dx \le 1.$$

Define the mapping $T\colon C([0,1]) \to C([0,1])$,

$$Tu(x) = \int_0^1 g(x-y)F\big(u(y)\big)\, dy \qquad \text{for } x \in [0,1].$$

Show that there exists a unique function $u \in C([0,1])$ such that

$$u(x) = Tu(x) \qquad \text{for all } x \in [0,1].$$

*You may assume without proof that* $Tu \in C([0,1])$ *for all* $u \in C([0,1])$.

**Problem 6.** (10 points)

Let $(U, \|\cdot\|_U)$ and $(V, \|\cdot\|_V)$ be normed spaces and let $T\colon U \to V$ be a bounded linear mapping. Show that $T$ is continuous.

**Problem 7.** (10 points)

Consider the mapping $T\colon \ell^2 \to \ell^2$ given by

$$(Tx)_n = \begin{cases} 2x_1 - x_2 & \text{if } n = 1, \\ 2x_n - x_{n-1} - x_{n+1} & \text{if } n \ge 2, \end{cases}$$

for $x = (x_1, x_2, \ldots) \in \ell^2$.

a) Show that $T\colon \ell^2 \to \ell^2$ is a bounded linear operator.

b) Find the adjoint $T^*$ of $T$.

**Problem 8.** (10 points)

Assume that $U$ is a Hilbert space over the real numbers $\mathbb{R}$. Recall that the space $L(U, U)$ of bounded linear operators $T\colon U \to U$ is a Banach space with the norm

$$\|T\|_{L(U,U)} = \sup_{\|u\|_U \leq 1} \|Tu\|_U.$$

Denote now by

$$\mathcal{S} := \{T \in L(U, U) : T \text{ is self-adjoint}\}$$

the set of self-adjoint bounded linear operators $T\colon U \to U$.

Show that $\mathcal{S}$ is a closed subspace of $\left(L(U, U), \|\cdot\|_{L(U,U)}\right)$.