Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods

Fall 2020

Exercise set 6

Please justify your answers! Note that *how* you arrive at an answer is more important than the answer itself.

1 **a)** Determine if the following expressions are norms on $\mathbb{R}^3$:

i) $f(x_1, x_2, x_3) = |x_1| + |x_2|$;

ii) $f(x_1, x_2, x_3) = |x_1| + (|x_2|^2 + |x_3|^2)^{1/2}$.

b) Show that if $\|\cdot\|$ is a norm on the scalar field $\mathbb{F}$, then there exists a positive number $\lambda > 0$ such that $\|x\| = \lambda |x|$ for all $x \in \mathbb{F}$.

c) Let X be a vector space and let $\|\cdot\|_a$ and $\|\cdot\|_b$ be norms on X. Show that

$$\|x\| := \left(\|x\|_a^2 + \|x\|_b^2\right)^{1/2}$$

defines a norm on X.

2 For fixed indices $1 \leq p < q \leq \infty$, prove the following statements:

a) $\ell^p \subseteq \ell^q$.

b) $\|x\|_q \leq \|x\|_p$ for every $x \in \ell^p$. Hint: Consider first sequences x with $\|x\|_\infty = 1$.

3 Find a sequence $x = (x_1, x_2, \ldots)$ of real numbers which converges to 0, but which is not in any space $\ell^p(\mathbb{R})$, $1 \leq p < \infty$.

4 Given a normed space $(X, \|\cdot\|)$, prove that the function $\rho : X \to \mathbb{R}$ defined by $\rho(x) = \|x\|$ is continuous on X. Note that, as a consequence, we have

$$x_n \to x \quad \Rightarrow \quad \|x_n\| \to \|x\|.$$

5 Let X be a normed space. Given a *subspace* M of X, prove the following statements:

a) The closure $\overline{M}$ of M is also a subspace of X.

b) If $M \neq X$, then the interior of M is empty ($M^\circ = \emptyset$).

6 Show that if $f \in C(\mathbb{R})$ is uniformly continuous and for each $n \in \mathbb{N}$ we set $f_n(t) = f(t - \frac{1}{n})$, then $f_n \to f$ in $C(\mathbb{R})$. Show by example that this can fail if $f \in C(\mathbb{R})$ is not uniformly continuous.