Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods
Fall 2020

Exercise set 4

Please justify your answers! Note that *how* you arrive at an answer is more important than the answer itself.

1 Show that every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X,d) is bounded.

2 Take $X = \mathbb{R}$, and find the interior E°, the boundary ∂E and the closure $\overline{E}$ for each of the following subsets:

a) $E = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \ldots \right\}$.

b) $E = [0,1)$.

c) $E = \mathbb{R} \setminus \mathbb{Q}$, the set of irrationals.

3 Let (X,d) be a *complete* metric space, and let E be a subset of X. Prove that E is closed if and only if (E,d) is complete.

4 Let E be a subset of a metric space (X,d). Prove that E is dense in X if and only if $E \cap U \neq \emptyset$ for every nonempty open set $U \subset X$.

5 Let c_{00} denote the subset of ℓ^∞ with only finitely many nonzero elements,

$$c_{00} = \{x = (x_1, \ldots, x_N, 0, 0, \ldots) : N > 0, x_1, \ldots, x_N \in \mathbb{R}\},$$

and let $c_0 \subset \ell^\infty$ be the subset of sequences tending to zero,

$$c_0 = \left\{ x = (x_k)_{k\in\mathbb{N}} : \lim_{k\to\infty} x_k = 0 \right\}.$$

a) Show that the closure of c_{00} in ℓ^∞ is c_0.

b) Show that c_0 is closed in (ℓ^∞, d_∞). Thus, we may conclude from Problem 3 that (c_0, d_∞) is a complete metric space.

6 Let (X, d) be a metric space.

a) Prove that every finite subset $E \subseteq X$ is compact.

b) Prove that every compact subset $E \subseteq X$ is bounded.

c) Let $X = \mathbb{R}$ and $E = \{\frac{1}{n}\}_{n \in \mathbb{N}}$. Provide an open cover of E which does not have a finite subcover (thus showing E is not compact).