Norwegian University of Science and Technology
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TMA4145 Linear Methods
Fall 2020

Exercise set 10

Please justify your answers! Note that *how* you arrive at an answer is more important than the answer itself.

1 Which of the following transformations are linear?

a) $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T(p)(x) = xp(x) + p'(x)$, where $P_n(\mathbb{R})$ denotes the vector space of real-valued polynomials of degree at most n.

b) $T : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $T(z_1, z_2) = (\overline{z_1}, \overline{z_2})$, where $\mathbb{C}^2$ is a vector space over $\mathbb{R}$.

Does the conclusion change if $\mathbb{C}^2$ is considered as a vector space over $\mathbb{C}$? Explain.

c) Let $M_{n\times n}(\mathbb{R})$ denote the space of all $n \times n$ matrices with real entries.

i) $T : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$, $T(A) = A^2$.

ii) $T : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$, $T(A) = \det A$.

2 Given normed spaces X and Y, and a bounded and linear operator $A : X \to Y$, prove that $\ker(A)$ is a closed subspace of X. See Problem 5 for an example that range(A) need not be closed in Y.

3 Let T be the integral operator

$$Tf(x) = \int_0^1 k(x, y)f(y)dy,$$

defined by a kernel $k \in C([0, 1] \times [0, 1])$ such that $k(x, y) \geq 0$ for any $(x, y) \in [0, 1] \times [0, 1]$. Show that the operator norm of T as a mapping on $C[0, 1]$ with respect to $\|.\|_\infty$-norm is

$$\|T\| = \max_{x\in[0,1]} \int_0^1 |k(x, y)|dy.$$

4 Let M be a closed subspace of a Hilbert space H, and let P be the orthogonal projection of H onto M. Prove that P is bounded and linear, and find $\|P\|$. Is P isometric?

5 Define an operator $B : \ell^1 \to \ell^1$ by

$$Bx = \left(\frac{x_k}{k}\right)_{k\in\mathbb{N}} = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\right), \quad x = (x_k)_{k\in\mathbb{N}} \in \ell^1.$$

a) Show that B is bounded and linear, and $\|B\| = 1$.

b) Show that B is injective, but not surjective.

c) Prove that range(B) is a proper dense subspace of ℓ^1, but it is *not closed*.