Systems of Differential Equations

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Systems of Differential Equations

CASE 1: Consider the differential equation (D.E)

$$y'(t) = py(t) \qquad ------ \qquad (1)$$

Then $y(t) = Ce^{pt}$ is a solution to (1).

CASE 2: Consider the system of D.E's

$$\mathbf{y}'(t) = A\,\mathbf{y}(t) + \mathbf{g}(t).$$

Particularly, we consider a system of D.E's that is

i. first order,

ii. linear,

iii. homogeneous (ie. $\mathbf{g}(t) = 0$), and has

iv. constant coefficients.

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Systems of Differential Equations

$$\mathbf{NB:}\quad \mathbf{y}'(t) = \begin{bmatrix} y_1'(t) \\ \vdots \\ y_n'(t) \end{bmatrix},\quad \mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix}, A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$\mathbf{Theorem:(S.S)}-$ The set of solutions to $\mathbf{y}'(t) = A\,\mathbf{y}(t)$ forms a vector space. That is, if $\mathbf{y}_1$ and $\mathbf{y}_2$ are solutions of the system, then $c_1\mathbf{y}_1 + c_2\mathbf{y}_2$ is a solution for all $c_1, c_2 \in \mathbb{R}$

Systems of Differential Equations

EXAMPLE 1

Consider

$$\begin{aligned} y_1'(t) &= -5y_1(t) \\ y_2'(t) &= 3y_2(t) \end{aligned}$$

$$\underbrace{\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix}}_{\mathbf{y}'} = \underbrace{\begin{bmatrix} -5 & 0 \\ 0 & 3 \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}}_{\mathbf{y}} \iff \mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{-5t} \\ c_2 e^{3t} \end{bmatrix}$$

$$\mathbf{y}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t}$$

Systems of Differential Equations

EXAMPLE 2

Consider the system

$$y_1'(t) = \ \ y_1(t) + 2y_2(t)$$
$$y_2'(t) = 3y_1(t) + 2y_2(t)$$

So $\mathbf{y}'(t) = A\,y(t)$; where $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, and $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ are eigenvectors corresponding

to the eigenvalues -1 and 4 respectively. So $P = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$ such that $A = PDP^{-1}$.

Systems of Differential Equations

$$\mathbf{y}' = A\,\mathbf{y}$$

$$P^{-1}\mathbf{y}' = P^{-1}A\,\mathbf{y}$$

$$(P^{-1}\mathbf{y})' = P^{-1}(PDP^{-1})\,\mathbf{y}$$

$$(P^{-1}\mathbf{y})' = DP^{-1}\,\mathbf{y}$$

$$\mathbf{x}' = D\,\mathbf{x}$$

where $\mathbf{x} = P^{-1}\mathbf{y}$ and $\mathbf{y} = P\mathbf{x}$.

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{4t} \end{bmatrix}$$

$$\mathbf{y}(t) = P\,\mathbf{x}(t) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{4t}$$

Systems of Differential Equations

Theorem : Let A be a diagonalizable $n \times n$ matrix. If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are n linearly independent eigenvectors with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$,

then $\mathbf{v}_1 e^{\lambda_1 t}, \ldots, \mathbf{v}_n e^{\lambda_n t}$ is a basis for the solution space of $\mathbf{y}'(t) = A\mathbf{y}(t)$. That is, $c_1 \mathbf{v}_1 e^{\lambda_1 t} + \cdots + c_n \mathbf{v}_n e^{\lambda_n t}$ is a general

solution of $\mathbf{y}'(t) = A\mathbf{y}(t)$.

Systems of Differential Equations

Q : What happens if an $n \times n$ matrix A is not diagonalizable?

A : We find the generalized eigenvector for the matrix A in the space $Nul(A - \lambda I)^r$, where r is the algebraic multiplicity of λ.

INITIAL VALUE PROBLEM

Definition : An initial value problem is a system $\mathbf{y}'(t) = A\mathbf{y}(t)$ together with an initial condition $\mathbf{y}(t_0) = \mathbf{y}_0$ where $t_0 \in \mathbb{R}$ and $\mathbf{y}_0$ is a vector in $\mathbb{R}^n$.

Question – 5 minutes

Solve the initial value problem

$$y_1'(t) = 3y_1(t) + \ y_2(t), \qquad \mathbf{y}(0) = \begin{bmatrix} 1 \\ -4 \end{bmatrix}.$$
$$y_2'(t) = 9y_1(t) - 5y_2(t)$$

Two-dimensional System & Phase Diagrams

Consider the system

$$\mathbf{y}'(t) = A\mathbf{y}(t)$$

and let $\mathbf{y}(t)$ be its solution. What is $\mathbf{y}(t)$ as $t \to \infty$?

CASE 1: Real Diagonalizable - Distinct real roots

$$\mathbf{y}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2}$$

$\lambda > 0 - - - e^{\lambda t}$ increases$- - -\mathbf{v}e^{\lambda t}$ tends away from 0

$\lambda = 0 - - - - - - e^{\lambda t}$ is constant$- - - \mathbf{v}e^{\lambda t}$ remain still

$\lambda < 0 - - - e^{\lambda t}$ decreases $- - - \mathbf{v}e^{\lambda t}$ tends towards 0

Question – 10 minutes

Sketch a phase diagram for the following systems of differential equations

i. $\mathbf{y}'(t) = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \mathbf{y}(t)$

ii. $\mathbf{y}'(t) = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{y}(t)$

iii. $\mathbf{y}'(t) = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \mathbf{y}(t)$

Two-dimensional System & Phase Diagrams

CASE 2: Complex (not real) Diagonalizable

$\textbf{Theorem}$: Suppose that $\alpha + i\beta,\ \beta \neq 0,$ is a complex eigenvalue of A and let $\mathbf{v}$ be its corresponding eigenvector. Then

$$\mathbf{y}_1(t) = e^{\alpha t}(Re(\mathbf{v})\cos(\beta t) - Im(\mathbf{v})\sin(\beta t))$$

and

$$\mathbf{y}_1(t) = e^{\alpha t}(Re(\mathbf{v})\cos(\beta t) - Im(\mathbf{v})\sin(\beta t))$$

is a basis for the real solution space of $\mathbf{y}' = A\mathbf{y}$.

$\textbf{NB :} \mathit{We\ are\ interested\ in\ obtaining\ real\ function\ solutions.}$

$\alpha > 0 - - - - -$ motions (trajectories) spiral outwards

$\alpha = 0 - - - - - - -$ motion s(trajectories) are circular

$\alpha < 0 - - - - --$ motion (trajectory) spirals inwards

Question - 5 minutes

Let $\mathbf{y}(t) = c_1 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + c_2 \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$ be the general solution of the system

$\mathbf{y}'(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{y}(t)$ with initial condition $\mathbf{y}(0) = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$.

i. Find c_1 and c_2.

ii. What is $\mathbf{y}(\frac{\pi}{4})$?

Two-dimensional System & Phase Diagrams

$\textbf{CASE 3 :}$ One (repeated) real root

Definition: A vector $\mathbf{v}$ is called a $generalized\ eigenvector\ of\ rank\ r$ of a matrix A and corresponding to the eigenvalue λ if

$$(A-\lambda I)^k\mathbf{v}=\mathbf{0} \quad \text{but} \quad (A-\lambda I)^{k-1}\mathbf{v}\neq\mathbf{0}.$$

Questions - 40 minutes

1. Solve the initial value problem

$$\mathbf{y}'(t) = \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{y}(t), \qquad \mathbf{y}(0) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$

2. Let $\mathbf{y}(t) = c_1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{t}$

   be the general solution to the system $\mathbf{y}'(t) = A\mathbf{y}(t)$.

   i. Find the matrix A.

   ii. Using any known theorem, show that A is necessarily diagonalizable.

   iii. Find the values of c_1, c_2, and c_3 such that the system $\mathbf{y}'(t) = A\mathbf{y}(t)$ satisfies $A = PDP^T$.