Interpolation, Regression, and Markov Chains

Instructions

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Quick RECAP

0.1. THEOREM: Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathbb{R}^n.$$

A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the column of the identity matrix in $\mathbb{R}^n$:

$$A = [T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)].$$

0.2. A is called the $\textbf{STANDARD MATRIX}$ for the linear transformation

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0.3. Let $T : \mathbb{P}_2 \to \mathbb{P}_3$ be a linear transformation defined by

$$T(f(x)) = (x+1)f(x).$$

Basis for $\mathbb{P}_2$ is $\mathcal{B} = \{1, x, x^2\}$ and Basis for $\mathbb{P}_3$ is $\mathcal{C} = \{1, x, x^2, x^3\}$.

The representation matrix A for T is given by $A = [\ [T(1)]_{\mathcal{C}}\ \ [T(x)]_{\mathcal{C}}\ \ [T(x^2)]_{\mathcal{C}}\ ]$.

$$T(1) = 1 + x, \quad T(x) = x + x^2, \quad T(x^2) = x^2 + x^3$$

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$$[T(1)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad [T(x)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad [T(x^2)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Hence, the representation matrix A for the transformation T is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Interpolation, Regression and Markov Chains

Let A be an $m \times n$ matrix, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^m$ and let $A\mathbf{x} = \mathbf{b}$.

1. The system of linear equations $A\mathbf{x} = \mathbf{b}$ is:

   i). Consistent - (has at least one solution) - if $\mathbf{b} \in$ Col $A \subset \mathbb{R}^m$.

   ii). Inconsistent - (has no solution) - if $\mathbf{b} \notin$ Col A.

2. i). What happens if $\mathbf{b} \notin$ Col A?

   ii). We can only approximate $\mathbf{b}$ using $A\mathbf{x}$.

DIAGRAM 1 - Consistent system - Unique solution

Consider the system of equations:

$$
\begin{aligned} x_1 - x_2 &= 1 \\ x_1 + x_2 &= 1 \end{aligned} \quad \text{and} \quad \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\mathbf{b}}. \tag{0.1}
$$

- $A \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\operatorname{Col} A = \operatorname{Sp}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} = \mathbb{R}^2$.
- $\hat{\mathbf{b}} = \mathrm{P}_{\operatorname{Col} A}\mathbf{b} = \mathbf{b} \in \operatorname{Col} A$

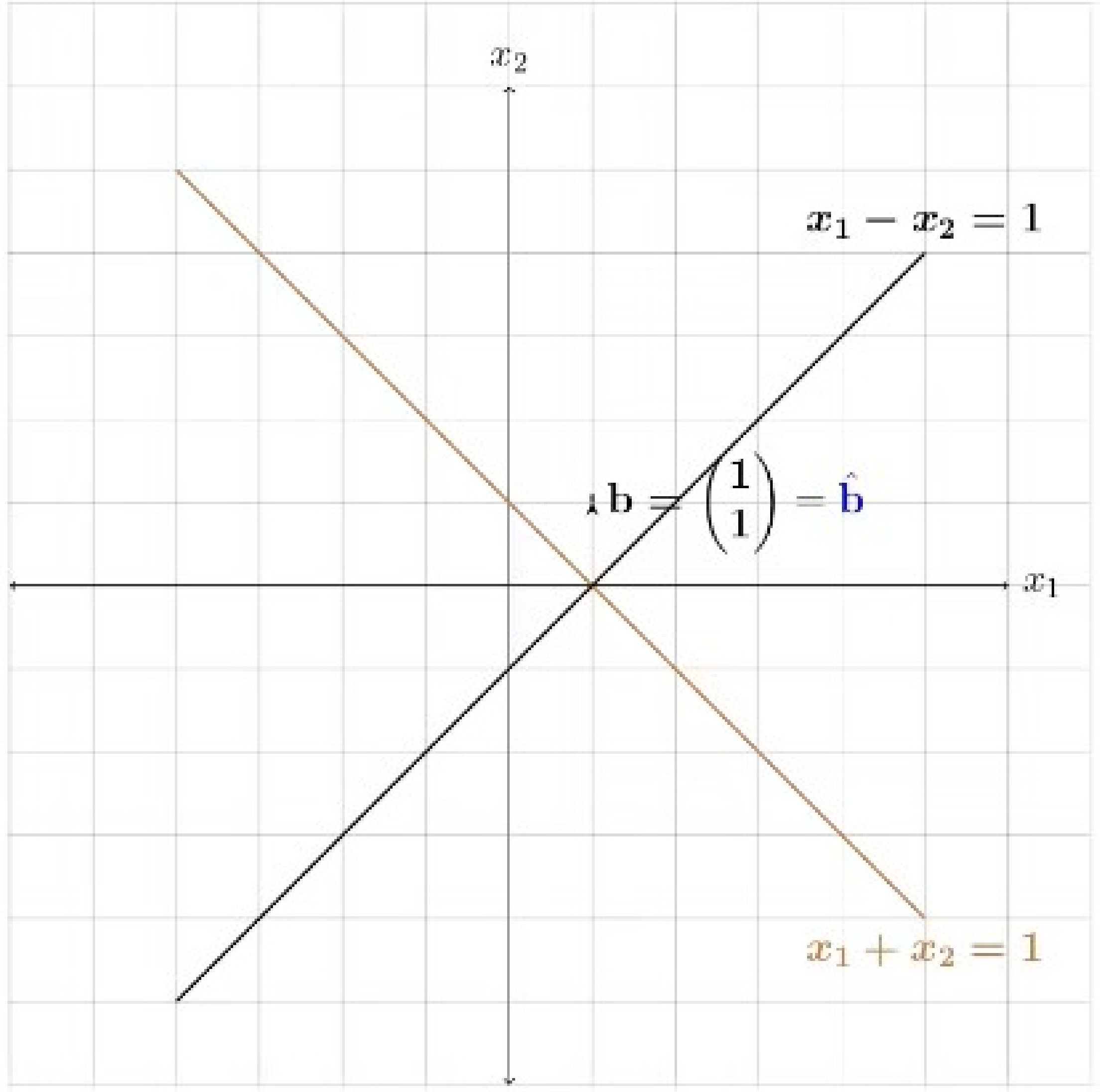


DIAGRAM 2 - Consistent system - Infinitely many solutions

Consider the system of equations:

$$(0.1)\qquad \begin{aligned} x_1 - x_2 &= 1 \\ -x_1 + x_2 &= -1 \end{aligned} \quad \text{and} \quad \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\mathbf{b}}.$$

- $(A|\mathbf{b}) \sim \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right)$ and $\mathrm{Col}\, A = \mathrm{Sp}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\} \neq \mathbb{R}^2$.
- $\hat{\mathbf{b}} = \mathrm{P}_{\mathrm{Col}\, A}\mathbf{b} = \mathbf{b} \in \mathrm{Col}\, A$

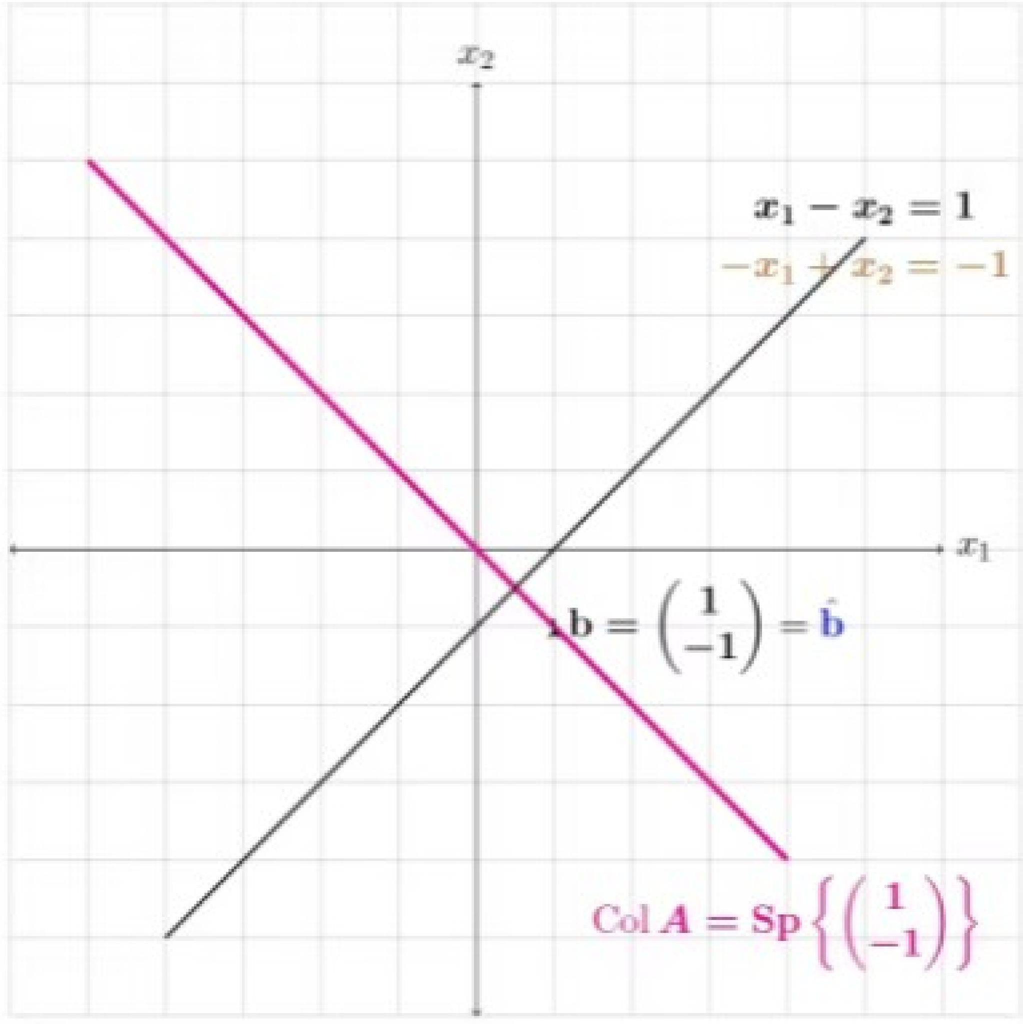


DIAGRAM 3 - Inconsistent system - No solution

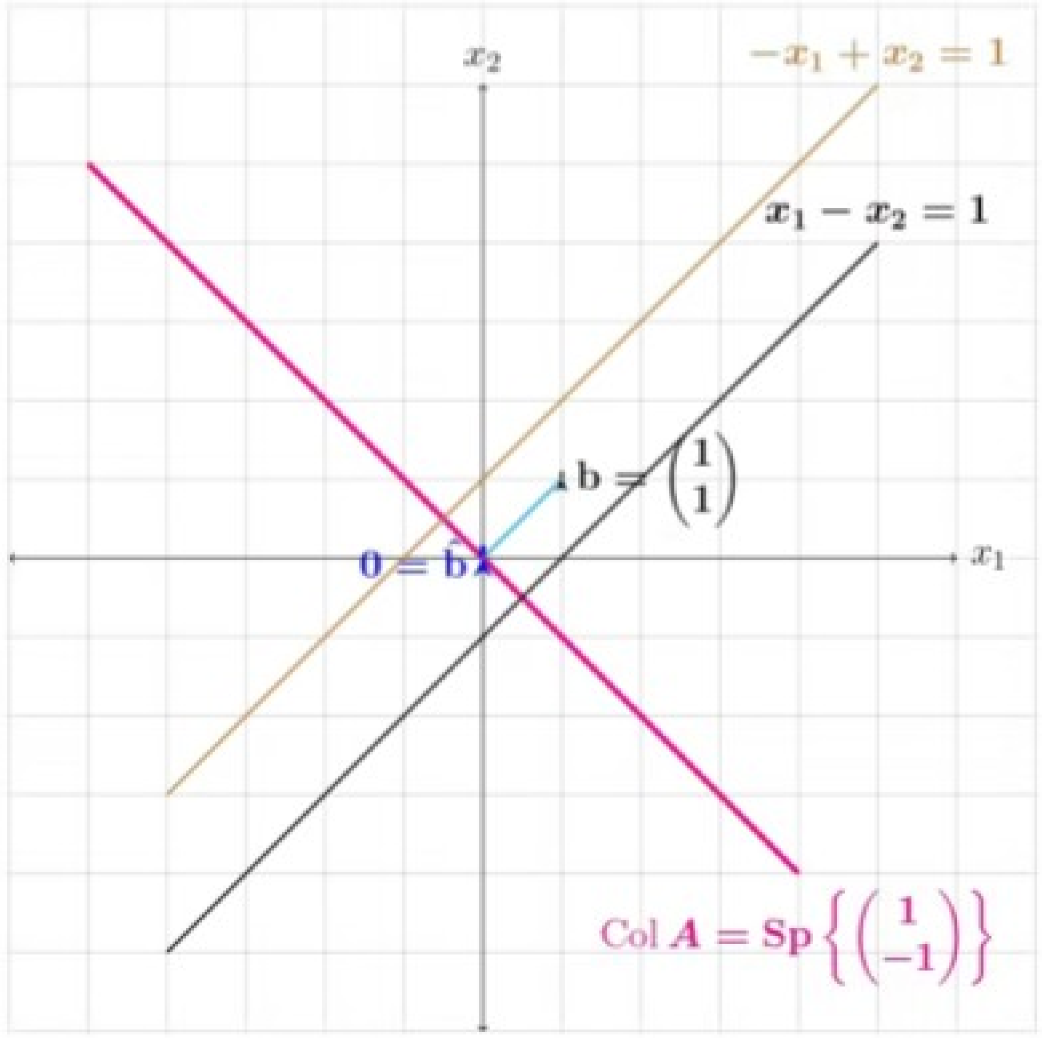
Consider the system of equations:

$$x_1 - x_2 = 1 \quad\quad (0.1)$$
$$-x_1 + x_2 = 1$$

and

$$\underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\mathbf{b}}.$$

- $(A|\mathbf{b}) \sim \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 2 \end{array}\right)$ and $\operatorname{Col} A = \operatorname{Sp}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\} \neq \mathbb{R}^2$.
- $\hat{\mathbf{b}} = \mathrm{P}_{\operatorname{Col} A}\mathbf{b} = \mathbf{0} \in \operatorname{Col} A$



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3. **Definition :** If A is an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$, then a least-squares solution of $A\mathbf{x} = \mathbf{b}$ is an element $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that

$$||\mathbf{b} - A\hat{\mathbf{x}}|| \leq ||\mathbf{b} - A\mathbf{x}||$$

for all $\mathbf{x} \in \mathbb{R}^n$.

4. By the **Orthogonal Decomposition :** $A^T(A\mathbf{x} - \mathbf{b}) = 0$.

5. **Normal Equations :** $A^T A\mathbf{x} = A^T\mathbf{b}$.

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6. **Theorem :** Every solution of the normal equation $A^T A\mathbf{x} = A^T\mathbf{b}$ is a least-squares solution of the equation $A\mathbf{x} = \mathbf{b}$.

7. Let A be an $m \times n$ matrix. Then the following are equivalent:

i). $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b} \in \mathbb{R}^m$.

ii). The matrix $A^T A$ is invertible.

iii). The columns of A are linearly independent.

Let A be an $m \times n$ matrix. Then the matrix $A^T A$ is always symmetric.

TRUE

FALSE

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8. **Probability vector :** A vector $\mathbf{v}$ with nonnegati ve entries that sum up to 1.

9. **Stochastic matrix :** A square matrix whose columns are probability vectors.

10. **Definition :** Let M be a stochastic matrix and $\mathbf{x}_0$ is a probability vector. For $n = 0, 1, 2, \ldots$, we call the sequence of vectors $\{\mathbf{x}_n\}$ a **Markov chain**.

Every unit vector is a probability vector.

TRUE

FALSE

Every probability vector is a unit vector.

✗ TRUE

✓ FALSE

The identity matrix is a stochastic matrix.

TRUE

FALSE

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11. $\textbf{Theorem :}$ A stochastic matrix M always has $\lambda = 1$ as an eigenvalue.

12. $\textbf{Equilibrium or steady} - \textbf{state vector :}$ Is a probability vector, which is also an eigenvector corresponding to the eigenvalue 1 of a stochastic matrix M.

13. $\textbf{Regular Stochastic Matrix :}$ A stochastic matrix M is called regular if there is a $k \geq 1$ such that all the elements in M^k are strictly positive (.i.e. > 0).

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14. **Theorem :** Let M be a regular stochastic matrix.Then M has a unique equilibrium vector $\mathbf{q}$.
For any initial probability vector $\mathbf{x}_0$, the Markov chain $\{\mathbf{x}_n\}$converges to $\mathbf{q}$
when $n \to \infty$.

A stochastic triangular matrix is regularly stochastic.

TRUE

FALSE

The matrix $M = \begin{pmatrix} 0.4 & 0.2 \\ 0.2 & 0.7 \\ 0.4 & 0.1 \end{pmatrix}$ is stochastic.

TRUE ✗ FALSE ✓

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The matrix $M = \begin{pmatrix} 0.4 & 0.2 & 1 \\ 0.2 & 0.7 & 0 \\ 0.5 & 0.1 & 0 \end{pmatrix}$ **is stochastic.**

✕ TRUE ✓ FALSE

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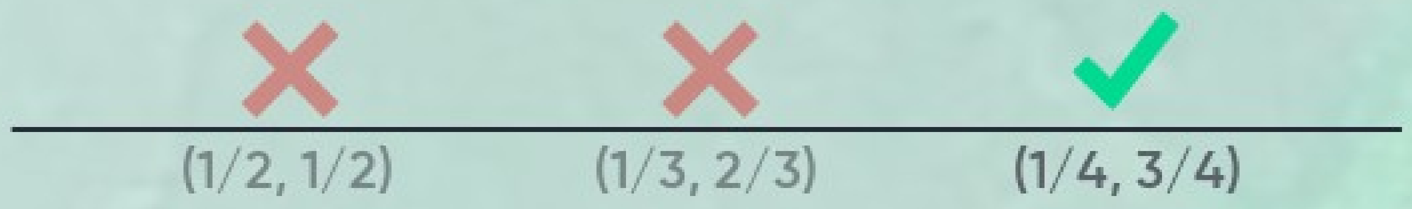
The matrix $M = \begin{pmatrix} 0 & 0.3 \\ 1 & 0.7 \end{pmatrix}$ is regularly stochastic.

TRUE

FALSE

Find the equilibrium or steady-state vector of $M = \begin{pmatrix} 0.1 & 0.3 \\ 0.9 & 0.7 \end{pmatrix}$.



QUESTIONS - 40 MINUTES

1. Use the least-squares method to find

i. the linear polynomial and

ii. the quadratic polynomial

that best fit the data points $(1,-2),(0,3),(0,1),(1,0)$, and $(-1,1)$.

2. Let the matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a stochastic matrix.

i. Show that M has at least one equilibrium or steady-state vector.

ii. Show that M can be written in the form $\begin{bmatrix} 1-c & b \\ c & 1-b \end{bmatrix}$, where $b, c \in (0,1)$.

iii. If $b = c = 0$, then prove that there are two linearly independent equilibrium vectors.

3. Decide whether or not each of the following matrices M is regularly stochastic. If yes, find the matrix M^k for some $k \geq 1$ whose entries are strictly positive, and find the equilibrium or steady-state vector of M.

i. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

ii. $\begin{bmatrix} 0 & 1 & 0.5 \\ 0.4 & 0 & 0 \\ 0.6 & 0 & 0.5 \end{bmatrix}$