TMA4110 Matematikk 3

Brief solutions to Assignment 1

Chapter 1

Exercise 1:

a) Use the quadratic formulas.

$$(1-i)^2 = 1^2 + i^2 - 2i = 1 - 1 - 2i = -2i$$

b) (HINT: Use that $w \cdot \bar{w} = |w|^2$.)
We start by computing the denominator of the fraction

$$2 - i + \overline{3+2i} = 2 - i + 3 - 2i = 5 - 3i.$$

Now using the above hint we get that

$$\begin{aligned}\frac{1+3i}{2-i+\overline{3+2i}} &= \frac{1+3i}{5-3i} \\ &= \frac{(1+3i)(5+3i)}{(5-3i)(5+3i)} \\ &= \frac{(1+3i)(5+3i)}{5^2+3^2} \\ &= \frac{5+3i+15i-9}{34} \\ &= \frac{-4+18i}{34} \\ &= \frac{-2}{17} + \frac{9}{17}i\end{aligned}$$

c) Use that $i^4 = (i^2)^2 = (-1)^2 = 1$ and that $\frac{1}{i} = -i$ since $i \cdot (-i) = -i^2 = 1$.

$$\frac{1}{i^5} = \frac{1}{i^4 \cdot i} = \frac{1}{i} = -i$$

Exercise 2:

a)

$$-1 + \overline{2+3i} = 1 - 3i = \sqrt{1^2 + (-3)^2} e^{\arctan(-3)}$$

Remember that the formular for the argument depends on which quadrant we are in.

PS $\arctan(-3) \simeq -1.24904577$ in radians. However, answering with the expression should be completely acceptable.

b) Compute the polarform of

$$\frac{1+\sqrt{3}i}{1-\sqrt{3}i}$$

and use the multiplication rules for polarforms.

First
$$\left\| \frac{1+\sqrt{3}i}{1-\sqrt{3}i} \right\| = 1$$
since the numerator and denominator are complex conjugate. Furthermore,

$$\begin{aligned}\arg\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right) &= \arg(1+\sqrt{3}i) - \arg(1-\sqrt{3}i) \\ &= \arg(1+\sqrt{3}i) + \arg(1+\sqrt{3}i) \\ &= 2 \cdot \frac{\pi}{3}\end{aligned}$$

since dividing by a complex number becomes subtraction when we take the argument. All in all we get

$$\left\| \left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^8 \right\| = 1^8 = 1$$

and

$$\arg\left(\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^8\right) = 8 \cdot \left(\frac{2\pi}{3}\right)$$

so the polar form is

$$e^{i(8 \cdot 2\pi/3)} = e^{i(4\pi/3)}$$

by reducing the modulus.

Exercise 3: Use the fact the equation

$$z^n = c$$

has solutions

$$\sqrt[n]{|c|} \cdot e^{ik\theta/n} \qquad k \in \{0, 1, \ldots, n\}.$$

With $\theta = \arg(c)$.

Exercise 4: Multiplication geometrically can be seen multiplying the lengths of the complex numbers and adding the angles. This can bee seen by the computation

$$\begin{aligned} zw &= re^{i\theta}se^{i\alpha} \\ &= sre^{i\theta}e^{i\alpha} \\ &= sre^{i(\theta+\alpha)} \end{aligned}$$

Exercise 5: Write $z = a + bi$ where $a, b \in \mathbb{R}$, then $\bar{z} = a - bi$. From direct computation we get

$$\begin{aligned} 1/2(z+\bar{z}) &= 1/2(a+bi+a-bi) \\ &= 1/2(a+a) \\ &= a. \end{aligned}$$

the other formula is done similarly.

Exercise 6:

a) Look at exercise 5.

b) We have that

$$z = \frac{-2i \pm \sqrt{-4-4(-1-i)}}{-1-i}.$$

c) Look for the hidden quadratic equation. We have that

$$\begin{aligned} (z+1)^4 = (z-1)^4 \Leftrightarrow \left(\frac{z+1}{z-1}\right)^4 &= 1 \\ \Leftrightarrow \left(\frac{z+1}{z-1}\right)^2 &= \pm 1. \end{aligned}$$

Where the last holds since the complex solutions to the equations $w^2 = 1$ are exactly $w = \pm 1$. We now work case by case. For 1: we have

$$\left(\frac{z+1}{z-1}\right)^2 = 1$$

if and only if $(z+1)^2 = (z-1)^2$, that is if and only if $z^2 + 2z + 1 = z^2 - 2z + 1$ which is equivalent to $2z = -2z$, that is if and only if $z = 0$. For -1: we get that the above is equivalent to $(z+1)^2 = -(z-1)^2$. Which is to say, $z^2 + 2z + 1 = -z^2 + 2z - 1$. Which is equivalent to $2z^2 = -2$ that is $z = \pm i$. So the solutions are $z \in \{0, i, -i\}$.

Exercise 7:

a) We use the hint. Given any polynomial P with real coefficents then $P(\overline{z}) = \overline{P(z)}$.

$$\begin{aligned} P(\overline{z}) &:= \sum_{0}^{deg(P)} a_n \overline{z}^n \\ &= \sum_{0}^{deg(P)} a_n \overline{z^n} \\ &= \sum_{0}^{deg(P)} \overline{a_n z^n} \\ &= \overline{\sum_{0}^{deg(P)} a_n z^n} \\ &= \overline{P(z)} \end{aligned}$$

Here the first equality follows by the multiplicativity of taking the conjugate ($\overline{x \cdot y} = \overline{x} \cdot \overline{y}$). The secound equality follows from the coeffiecent a_n being real hence their own complex conjugate and the multiplicativity of taking the conjugate. And the third equality follows from conjugation commuting with addition ($\overline{x+y} = \overline{x} + \overline{y}$).

Now the result follows since assuming $P(z) = 0$ we have that $P(\overline{z}) = \overline{P(z)} = \overline{0} = 0$

b) By theorem 1.11 P must have three roots. i.e $P(z) = \prod_{i=1}^{3}(z - z_i)$. Remember that $\overline{z} = z$ if and only if z is real. If all roots of P are real then at least one of them are and we are done. Therefore assume not all roots of P are real. Pick a non-real root and denote it as z_1, by a) we now know that $\overline{z_1}$ is also a root and it is not equal to z_1. We choose to lable $z_2 := \overline{z_1}$. We know that d is real since P only has real coefficents. Furthermore $d = -z_1 z_2 z_3 = -z_1 \overline{z_1} z_3 = -|z_1|^2 z_3$. Since $-|z_1|^2$ is real and non-zero by our assumption that z_1 was non-real we have that $\frac{d}{-|z_1|^2}$ is real and $\frac{d}{-|z_1|^2} = z_3$. Hence, if we assume that P has at least one non-real root, it must have one real root and if P has no non-real roots then all of its roots are real.

Chapter 2

Exercise 8:

a) Start with $r_1 \longleftrightarrow r_3$ then work from top left down fixing a_{11}.

b) Same as above

Exercise 9: Note that the coefficient systems agree and perform row reduction on $[A|b_1, b_2]$.

Exercise 10: Row reduction gives us that

$$[A|b_1, b_2] \sim \left(\begin{array}{cccc|cc} 1 & 0 & 0 & -3/2 & -1/2 & -9/2 \\ 0 & 1 & 0 & 1/5 & 1/5 & 0 \\ 0 & 0 & 1 & 3/10 & 33/10 & 7/2 \end{array}\right)$$

so the claim is true.

Exercise 11:

a) Use row reduction.

b) Remember that you may multiply by complex numbers.

Exercise 12: They are both right. If you set $s' = -s + 3$ and $t' = -t + 4$ you see that one formula is just a linear shift of another formula. To see this, note that changing s is the only way to change the 2nd vector coordinate, and like wise varying t is the only way to change the 3rd vector coordinate.

Exercise 13: Assuming that $b \neq 0$ we can apply row reduction to get

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -ac - c\frac{a}{b} \\ 0 & 1 & 0 & c - \frac{c}{b} \\ 0 & 0 & 1 & \frac{c}{b} \end{array}\right)$$

Hence if $b \neq 0$ there is one solution given as follows

$$x_1 = -ac - c\frac{a}{b}, x_2 = c - \frac{c}{b}, x_3 = \frac{c}{b}$$

.

Assuming that $b = 0$ we observe from the bottom row that if $c \neq 0$ there is no solution. Assuming that $c = 0$ we now have to solve the following problem

$$\left(\begin{array}{ccc|c} 1 & a & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We do row reduction to get

$$\left(\begin{array}{ccc|c} 1 & 0 & -a & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We now have on free variable. Letting $t = x_2$ we find that $x_3 = -t$ and $x_1 = -at$.