EXTRA TASKS

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\-2\\1\end{bmatrix},\ T\left(\begin{bmatrix}-1\\2\\0\end{bmatrix}\right) = \begin{bmatrix}1\\4\\1\end{bmatrix} \text{ and } T\left(\begin{bmatrix}1\\3\\-1\end{bmatrix}\right) = \begin{bmatrix}3\\1\\5\end{bmatrix}.$$

i. Find the standard matrix A corresponding to the transformation T.

ii. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the columns of A. Show that the columns of A form an orthogonal set and find the orthogonal projection of the vector $\mathbf{u} = \begin{bmatrix}2\\1\\3\end{bmatrix}$ to Span$\{\mathbf{v}_1, \mathbf{v}_3\}$.

SOLUTION

i. The standard matrix A is given by

$$A = \left[\, T(\vec{e}_1) \quad T(\vec{e}_2) \quad T(\vec{e}_3) \,\right],$$

where $\vec{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$.

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\-2\\1\end{bmatrix}_{//},\quad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = ?\quad,\quad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = ?$$

- $$T\left(\begin{bmatrix}-1\\2\\0\end{bmatrix}\right) = T\left(-\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = -T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) + 2\,T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$$

$$\Rightarrow \; T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \frac{1}{2}\, T\left(\begin{bmatrix}-1\\2\\0\end{bmatrix}\right) + \frac{1}{2}\, T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix}1\\4\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\-2\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} \;//$$

- $$T\left(\begin{bmatrix}1\\3\\-1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\0\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\\0\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) + 3\,T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) - T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

$$\Rightarrow \; T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) + 3\,T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\3\\-1\end{bmatrix}\right)$$

$$= \begin{bmatrix}1\\-2\\1\end{bmatrix} + 3\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}3\\1\\5\end{bmatrix} = \begin{bmatrix}1\\0\\-1\end{bmatrix} \;//$$

So the standard matrix A is

$$A = \begin{bmatrix} \underbrace{\begin{matrix}1\\-2\\1\end{matrix}}_{\vec{v}_1} & \underbrace{\begin{matrix}1\\1\\1\end{matrix}}_{\vec{v}_2} & \underbrace{\begin{matrix}1\\0\\-1\end{matrix}}_{\vec{v}_3} \end{bmatrix}.$$

ii. Claim : $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set.

We will show that

$$\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1 \cdot \vec{v}_3 = \vec{v}_2 \cdot \vec{v}_3 = 0.$$

$$\vec{v}_1 \cdot \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 - 2 + 1 = 0 \ ; \ \vec{v}_1 \cdot \vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1 + 0 - 1 = 0$$

$$\vec{v}_2 \cdot \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1 + 0 - 1 = 0.$$ Hence, the above claim holds.

Let $W = \operatorname{Span}\{\vec{v}_1, \vec{v}_3\}$. Then the orthogonal projection of $\vec{y} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ onto W is given by $P_W \vec{y} = \dfrac{\vec{v}_1 \cdot \vec{y}}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 + \dfrac{\vec{v}_3 \cdot \vec{y}}{\vec{v}_3 \cdot \vec{v}_3} \vec{v}_3$.

$$\Rightarrow \vec{v}_1 \cdot \vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 6 \quad \text{and} \quad \vec{v}_1 \cdot \vec{y} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 3$$

$$\Rightarrow \vec{v}_3 \cdot \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 2 \quad \text{and} \quad \vec{v}_3 \cdot \vec{y} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = -1$$

$$P_W \vec{y} = \frac{3}{6} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} //$$

2. Show in each of the following cases whether or not the subsets of the given vector spaces are subspaces.

i. $U_1 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \middle| \; 2a - 3b = 0 \text{ and } a, b \in \mathbb{R} \right\} \subseteq \mathbb{R}^2$

ii. $U_2 = \left\{ \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \middle| \; ad = 0 \text{ and } a, c, d \in \mathbb{R} \right\} \subseteq M_{2\times 2}$, where $M_{2\times 2}$ is the vector space of all 2×2 matrices.

iii. $U_3^m = \left\{ \begin{bmatrix} 6s - 4t \\ 2s + t \\ t - m \end{bmatrix} \middle| \; s, t \in \mathbb{R} \right\} \subseteq \mathbb{R}^3$

a. when $m = 0$ and

b. when $m = 1$.

SOLUTION

i. Let $U_1 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \middle| \; 2a - 3b = 0 \text{ and } a, b \in \mathbb{R} \right\} \subseteq \mathbb{R}^2$.

- $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in U_1$ since $2(0) - 3(0) = 0$ and $0 \in \mathbb{R}$.

- Let $\vec{x}_1 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$ be in U_1. So

$$2a_1 - 3b_1 = 0 \quad \text{and} \quad 2a_2 - 3b_2 = 0.$$

$\vec{x}_1 + \vec{x}_2 = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \end{bmatrix}$. $\Rightarrow 2(a_1 + a_2) - 3(b_1 + b_2) = 2a_1 - 3b_1 + 2a_2 - 3b_2$

$\Rightarrow 2(a_1 + a_2) - 3(b_1 + b_2) = 0$

$\Rightarrow \vec{x}_1 + \vec{x}_2 \in U_1$.

- Let $\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix} \in U_1$ and $k \in \mathbb{R}$. Then $2a - 3b = 0$.

$$c\vec{x} = \begin{bmatrix} ca \\ cb \end{bmatrix}. \quad \Rightarrow \quad 2(ca) - 3(cb) = c\,(2a-3b) = 0.$$

$$\Rightarrow c\vec{x} \in U_1.$$

ii. $U_2 = \left\{ \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \;\middle|\; ad = 0 \text{ and } a, c, d \in \mathbb{R} \right\}.$

- Clearly, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in U_2$ since $ad = 0$ holds.

- U_2 is not closed under addition. We give a counterexample:

  Let $A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$. Clearly, $A_1 \in U_2$ and

  $A_2 \in U_2$ since $(0)(-1) = 0$ and $(2)(0) = 0$. But

$$A_1 + A_2 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad 2(-1) \neq 0.$$

Hence U_2 is NOT a subspace of $M_{2\times 2}$.

iii. a). Let $m = 0$.

$$U_3^0 = \left\{ \begin{bmatrix} 6s - 4t \\ 2s + t \\ t \end{bmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}$$

$$\Rightarrow \begin{bmatrix} 6s-4t \\ 2s+t \\ t \end{bmatrix} = \begin{bmatrix} 6s \\ 2s \\ 0 \end{bmatrix} + \begin{bmatrix} -4t \\ t \\ t \end{bmatrix} = s\begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}, \quad s,t \in R.$$

So U_3^0 is $\text{Span}\left\{\begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}\right\}$ and so U_3^0 is a subspace of $\mathbb{R}^3$.

b). Let $m=1$.

$$U_3^1 = \left\{\begin{bmatrix} 6s-4t \\ 2s+t \\ t-1 \end{bmatrix} \,\middle|\, t,s \in \mathbb{R}\right\}.$$

Claim: The zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not in U_3^1. So consider

$$\Rightarrow \begin{bmatrix} 6s-4t \\ 2s+t \\ t-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 6 & -4 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 6 & -4 & 0 \\ 0 & 7/3 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 6 & -4 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -7/3 \end{array}\right]$$

Since the system has NO solution, it implies that $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin U_3^1$.

U_3^1 is NOT a subspace of $\mathbb{R}^3$.

3. Let $C[0, \pi/2]$ be the vector space of continuous functions defined on $[0, \pi/2]$ with inner product

$$\langle f, g \rangle = \int_0^{\frac{\pi}{2}} f(t)g(t)\ dt,$$

where $f, g \in C[0, \pi/2]$. Let $W = \text{Span}\{1, \sin t\}$.

i. Is the set $\{1, \sin t\}$ an orthogonal set? Justify your answer.

ii. Find an orthogonal basis for W.

SOLUTION

i. Let $f(t) = 1$ and $g(t) = \sin t$.

$$\langle f, g \rangle = \int_0^{\pi/2} f(t) g(t)\, dt$$

$$= \int_0^{\pi/2} \sin t\, dt = \Big[-\cos t\Big]_0^{\pi/2} = -\Big[\cos \pi/2 - \cos 0\Big]$$

$\langle f, g \rangle = 1 \neq 0.$

So f and g are NOT orthogonal.

ii. Using the Gram-Schmidt orthogonalization process we will change $\{f, g\}$ to $\{h_1, h_2\}$ such that h_1 and h_2 are orthogonal.

Let $h_1(t) = f(t) = 1.$

$\Rightarrow \quad h_2(t) = g(t) - P_{h_1(t)} g(t)$, where $P_{h_1} g$ is the projection of g onto $h_1 = f$.

$$P_{h_1(t)} g(t) = \frac{\langle h_1, g \rangle}{\langle h_1, h_1 \rangle} h_1 = \frac{\langle f, g \rangle}{\langle f, f \rangle} f.$$

$\Rightarrow \langle f, g \rangle = 1$

$$\Rightarrow \langle f, f \rangle = \int_0^{\pi/2} dt = \Big[t \Big]_0^{\pi/2} = \pi/2$$

$$P_{h_1} g = \frac{1}{\pi/2}(1) = \frac{2}{\pi} \quad \text{and} \quad h_2(t) = \sin t - \frac{2}{\pi}$$

$$\langle h_1, h_2 \rangle = \int_0^{\pi/2} \left(\sin t - \frac{2}{\pi} \right) dt = \int_0^{\pi/2} \sin t \, dt - \frac{2}{\pi} \int_0^{\pi/2} dt$$

$$= 1 - \frac{2}{\pi}\left(\pi/2\right) = 0.$$

Hence, the set $\left\{ 1, \sin t - \frac{2}{\pi} \right\}$ is an orthogonal basis for W.

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - 3x_2 \\ -6x_1 + 9x_2 \end{bmatrix}.$$

i. Is T injective?

ii. Is T surjective?

SOLUTION

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - 3x_2 \\ -6x_1 + 9x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & -3 \\ -6 & 9 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -3 \\ -6 & 9 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix}.$$

$$\text{Col } A = \text{Span}\left\{\begin{bmatrix} 2 \\ -6 \end{bmatrix}\right\} \neq \mathbb{R}^2 \quad \text{i.e. } \text{Im } T \neq \mathbb{R}^2.$$

$$\text{Nul } A = \text{Span}\left\{\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right\} \neq \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}. \quad \text{i.e. } \text{Ker } T \neq \{\vec{0}\}.$$

i. T is NOT injective $\text{Ker } T \neq \{\vec{0}\}$.

ii. T is NOT surjective $\text{Im } T \neq \mathbb{R}^2$.

5. Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + 2x_3 \\ 2x_1 + x_2 + x_3 \end{bmatrix}.$$

i. Is S injective?

ii. Is S surjective?

SOLUTION

$$S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + 2x_3 \\ 2x_1 + x_2 + x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\text{Col } A = \text{Span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} = \mathbb{R}^2 \quad \text{i.e. } \operatorname{Im} S = \mathbb{R}^2$$

$$\text{Nul } A = \text{Span}\left\{\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}\right\} \quad \text{i.e. } \operatorname{Ker} S = \left\{ t\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\} \neq \{\vec{0}\}.$$

i. S is NOT injective since $\operatorname{Ker} S \neq \{\vec{0}\}$.

ii. S is surjective since $\operatorname{Im} S = \mathbb{R}^2$.

6. Find the general solution of the system

$$\begin{array}{rrrcl} (i-1)x_1 & -4x_2 & & = & 4 \\ ix_1 & -2ix_2 & +(2i+1)x_3 & = & i+1 \\ (3i+1)x_1 & & +(2i-1)x_3 & = & 5. \end{array}$$

SOLUTION

The matrix equation is given by

$$\begin{bmatrix} i-1 & -4 & 0 \\ i & -2i & 1+2i \\ 1+3i & 0 & -1+2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ i+1 \\ 5 \end{bmatrix}.$$

Consider the augmented matrix and find the reduced Echelon of the above matrix.

$$\left[\begin{array}{ccc|c} i-1 & -4 & 0 & 4 \\ i & -2i & 1+2i & 1+i \\ 1+3i & 0 & -1+2i & 5 \end{array}\right] \xrightarrow{\frac{1}{i-1}R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 2+2i & 0 & -2-2i \\ i & -2i & 1+2i & 1+i \\ 1+3i & 0 & -1+2i & 5 \end{array}\right]$$

$$\xrightarrow[R_3-(1+3i)R_1 \to R_3]{R_2 - iR_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 2+2i & 0 & -2-2i \\ 0 & 2-4i & 1+2i & -1+3i \\ 0 & 4-8i & -1+2i & 1+8i \end{array}\right] \xrightarrow{R_3 - 2R_2 \to R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 2+2i & 0 & -(2+2i) \\ 0 & 2-4i & 1+2i & -1+3i \\ 0 & 0 & -3-2i & 3+2i \end{array}\right] \xrightarrow{-\frac{1}{3+2i}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 2+2i & 0 & -(2+2i) \\ 0 & 2-4i & 1+2i & -1+3i \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{R_2-(1+2i)R_3 \to R_2} \left[\begin{array}{ccc|c} 1 & 2+2i & 0 & -(2+2i) \\ 0 & 2-4i & 0 & 5i \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[R_1-(2+2i)R_2 \to R_1]{\frac{1}{2-4i}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1-i \\ 0 & 1 & 0 & -1+\frac{i}{2} \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\begin{cases} x_1 = 1-i \\ x_2 = -1+\frac{i}{2} \\ x_3 = -1 \end{cases}$$