

Assume  $S$  admits calculus of right fractions,

pair  $X \xleftarrow{s} Y \xrightarrow{\alpha} Z$  with  $s \in S$

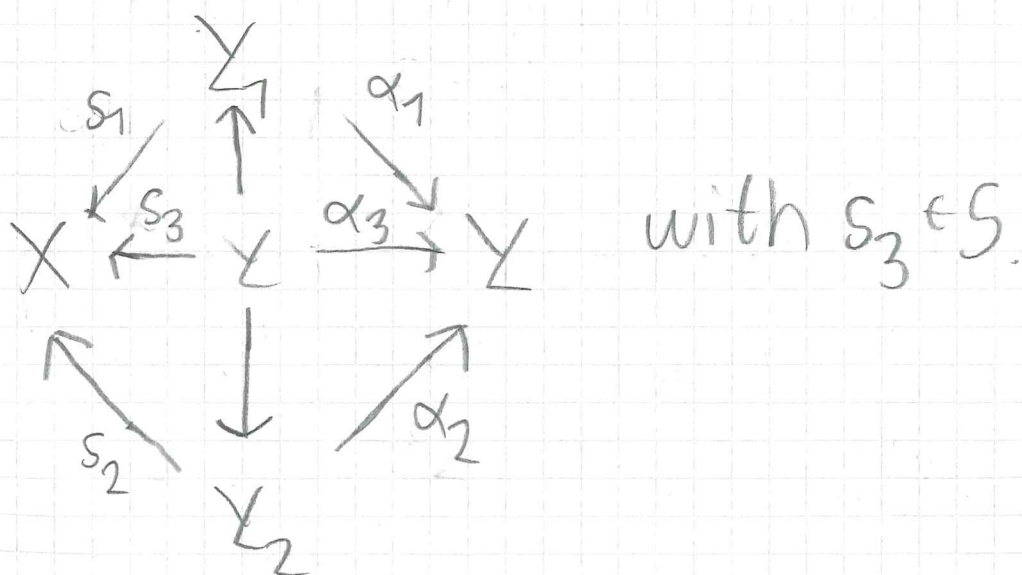
called right fraction,

Define cat  $S^{-1}\mathcal{C}$ :

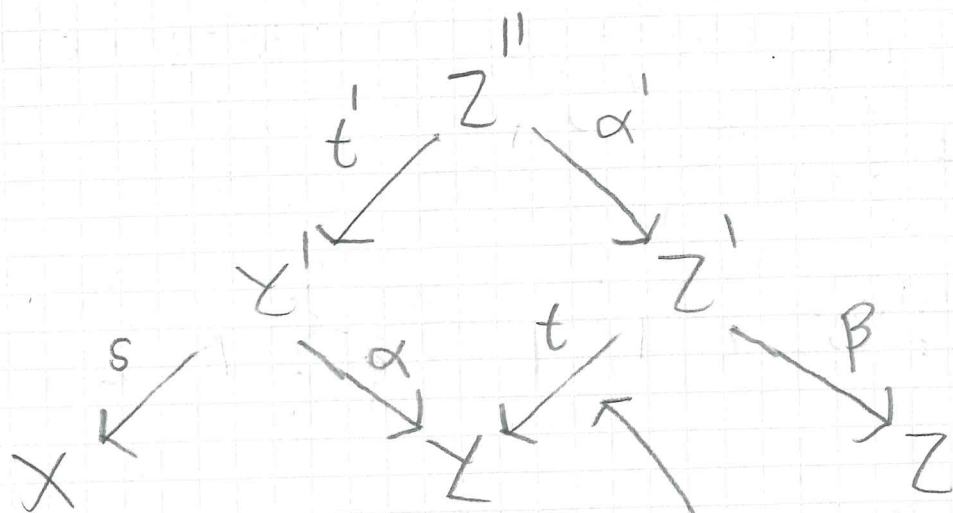
$Ob S^{-1}\mathcal{C} = Ob \mathcal{C}$

$Mor_{S^{-1}\mathcal{C}}(X, Y) =$  equivalence classes of right fractions  $X \xleftarrow{s} Y \xrightarrow{\alpha} Z$ , where

$(s_1, \alpha_1) \sim (s_2, \alpha_2)$  if  $\exists$  comm diagram



(Composite of  $[s, \alpha]$  and  $[t, \beta]$ ) is  $[s \circ t, \beta \circ \alpha]$  where



and where comm square obtained from (RF2).

Check (A bit of work)

- (1)  $\sim$  is an equivalence relation
- (2) composition in  $S^{-1}\mathcal{C}$  is well-defined
- (3)  $S^{-1}\mathcal{C}$  is a category (except perhaps that the Hom-spaces may not be sets)
- (4) Have isomorphism of categories

$F: S^{-1}\mathcal{C} \xrightarrow{\cong} \mathcal{C}[S^{-1}]$  given by identity on objects, and sending right fraction

$$X \xleftarrow{s} Y' \xrightarrow{\alpha} Y \text{ to } [X, \alpha, s, Y]$$

Back to complexes:  $\mathcal{A}$  abelian.

Want to localize at quasi-iso's, i.e. morphisms  $X \xrightarrow{f} Y$  s.t.  $H^n(f)$  iso  $\forall n \in \mathbb{Z}$ .

Problem:  $\{\text{quasi-iso's}\} \subset \text{Mor Ch}(\mathcal{A})$  does not admit a calculus of right (or left) fractions.

Solution: Consider  $K(\mathcal{A})$  instead of  $\text{Ch}(\mathcal{A})$

Proposition:  $S = \{[f] \mid f \text{ quasi-iso in } \text{Ch}(\mathcal{A})\} \subset \text{Mor } K(\mathcal{A})$  admits a calculus of right and left fractions.

Pf: (RF1) clear

(RF2) & (RF2)<sup>op</sup> - see Lemma 34.3 in notes (One condition)

(RF3) since  $K(\mathcal{A})$  additive, suffices to show:

$$X \xrightarrow{h} Y \xrightarrow{s} Y' \quad \text{so } h=0, \quad s \text{ quasi-iso}$$

$$\Rightarrow \exists \text{ quasi-iso } t: Y' \rightarrow X \text{ s.t. } h \circ t = 0$$

Assume  $h$  is as above

Apply  $\text{Hom}_{K(\mathcal{A})}(X, -)$  to triangle

$$Y \xrightarrow{s} Y' \xrightarrow{r} \text{Con}(s) \xrightarrow{\ell} Y[1]$$

$\text{RF-10-}$                        $\text{so-}$                        $\text{ko-}$

$$(X, \text{Con}(s)[1]) \rightarrow (X, Y) \rightarrow (X, Y') \rightarrow (X, \text{Con}(s)) \rightarrow \dots$$

$$\text{so } h=0 \Rightarrow \exists h': X \rightarrow \text{Cone}(s)[-1] \text{ s.t.}$$

$$\begin{array}{ccc} X & & \\ h' \downarrow & \searrow h & \\ \text{Cone}(s)[-1] & \xrightarrow{e[-1]} & X \end{array} \quad \text{commutes.}$$

$$\text{Set } \text{Cone}(h')[-1] = X' \xrightarrow{\text{triangle}} \text{get } X \xrightarrow{t} X \xrightarrow{h'} \text{Cone}(s)[-1] \rightarrow X'$$

$X' \xrightarrow{t} X$  quasi-iso since  $\text{Cone}(t) \cong \text{Cone}(s)[-1]$  in  $K(\mathcal{A})$ , and  $\text{Cone}(s)[-1]$  is exact.

Finally,  $h$  factors through  $\text{Cone}(s)[-1]$   
 $\Rightarrow h \circ t = 0$ . This shows (RF3). (RF3)<sup>op</sup> shown dually. ■

Definition:  $\mathcal{A}$ -abelian cat.

The derived category of  $\mathcal{A}$ , denoted  $D(\mathcal{A})$  is the localization of  $K(\mathcal{A})$  w.r.t class of quasi-isomorphisms,  $S^{-1}K(\mathcal{A})$

Remarks ·  $\text{Ob } D(\mathcal{A}) = \text{Ob } K(\mathcal{A}) = \text{Ob } \text{Ch}(\mathcal{A})$   
 chain complexes

$\text{Mor } D(\mathcal{A})$  - right fractions of morphisms in  $K(\mathcal{A})$ .

•  $D(\mathcal{A})$  may not be a cat, if Hom-spaces<sup>17</sup> are not sets. Need

$\text{Hom}_{D(\mathcal{A})}(X, Y) = \{\text{right fractions from } X \text{ to } Y\}$   
to be a set  $\forall X, Y \in D(\mathcal{A})$ . This holds in many cases.

e.g. if  $\mathcal{A} = \text{Mod } R$ ,  $R$  ring. We always assume this holds.  
What properties does  $D(\mathcal{A})$  have?

Lemma:  $\mathcal{B}$  additive cat,  $S \subset \text{Mor } \mathcal{B}$  admits calculus of right fractions. Then

(i)  $S^{-1}\mathcal{B}$  is additive

(ii) The functor  $\mathcal{B} \xrightarrow{q} S^{-1}\mathcal{B}$  is additive

Pf Define addition as follows:

Given  $(X \xleftarrow{s_1 \in S} X_1 \xrightarrow{\alpha} Y) \in \text{Mor } S^{-1}\mathcal{B}$

$(X \xleftarrow{s_2 \in S} X_2 \xrightarrow{\beta} Y) \in \text{Mor } S^{-1}\mathcal{B}$ .

want to add  $[s_1, \alpha]$  and  $[s_2, \beta]$ .

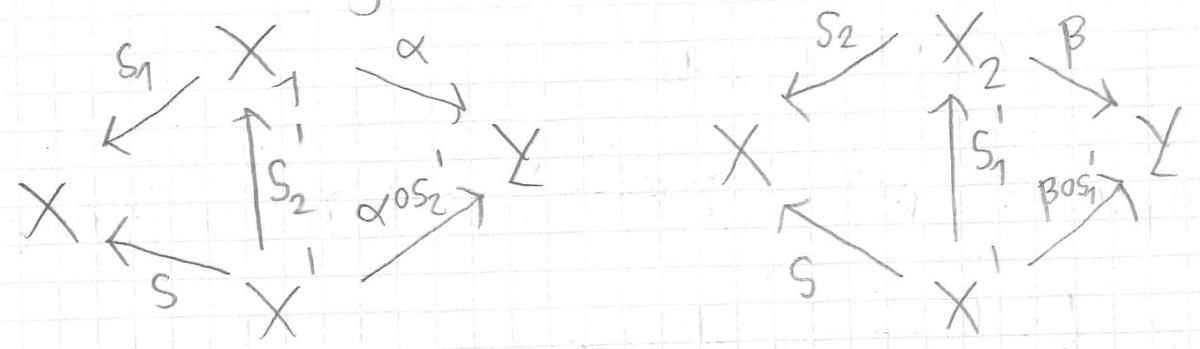
By (RF2), can complete diagram

$$\begin{array}{ccc} X' & \xrightarrow{s_1' \in S} & X_2 \\ s_2' \downarrow & & \downarrow s_2 \\ X_1 & \xrightarrow{s_1} & X \end{array}$$

Set  $s = s_2 s_1' = s_1 s_2'$

$S$  closed under composition  
 $\Rightarrow s \in S$

Have comm diagrams



$\Rightarrow [s_1, \alpha] = [s, \alpha \circ s_2']$  &  $[s_2, \beta] = [s, \beta \circ s_1']$   
 "common denominator"

$$[s_1, \alpha] + [s_2, \beta] = [s, \alpha \circ s_2'] + [s, \beta \circ s_1']$$

$$= [s, \alpha \circ s_2' + \beta \circ s_1']$$

Check: This is well-defined

Check: This makes  $\text{Hom}_{\mathcal{S}\mathcal{B}}^{-1}(X, Y)$  into an abelian group, with neutral element  $(X \xleftarrow{1} X \xrightarrow{0} Y)$

Check: composition in  $\mathcal{S}\mathcal{B}^{-1}$  is bilinear  
 $\Rightarrow \mathcal{S}\mathcal{B}^{-1}$  is preadditive.

Clearly, with this definition, the functor  $\mathcal{B} \xrightarrow{q} \mathcal{S}\mathcal{B}^{-1}$  is additive.

remains to show:  $\mathcal{S}\mathcal{B}^{-1}$  has a zero obj  
 - has biproducts.