Assume S admits calculus of right fractions.

pair $X \xleftarrow{s} Y' \xrightarrow{\alpha} Y$ with $s \in S$

called <u>right fraction</u>,

Define cat $S^{-1}\mathcal{C}$:

$\mathrm{Ob}\, S^{-1}\mathcal{C} = \mathrm{Ob}\,\mathcal{C}$

$\mathrm{Mor}_{S^{-1}\mathcal{C}}(X, Y)$ = equivalence classes of right fractions $X \xleftarrow{s} Y' \xrightarrow{\alpha} Y$, where

$(s_1, \alpha_1) \sim (s_2, \alpha_2)$ if $\exists$ comm diagram

$$\begin{tikzcd}
 & Y_1 \arrow[dl, "s_1"'] \arrow[dr, "\alpha_1"] & \\
X & Y' \arrow[l, "s_3"'] \arrow[r, "\alpha_3"] \arrow[u] \arrow[d] & Y \\
 & Y_2 \arrow[ul, "s_2"] \arrow[ur, "\alpha_2"'] &
\end{tikzcd}$$

with $s_3 \in S$.

Composite of $[s, \alpha]$ and $[t, \beta]$ is $[s \circ t', \beta \circ \alpha']$ where

$$\begin{tikzcd}
 & & Z'' \arrow[dl, "t'"'] \arrow[dr, "\alpha'"] & & \\
 & Y' \arrow[dl, "s"'] \arrow[dr, "\alpha"] & & Z' \arrow[dl, "t"'] \arrow[dr, "\beta"] & \\
X & & Y & & Z
\end{tikzcd}$$

and where comm square obtained from (RF2).

<u>Check (A bit of work)</u>

(1) $\sim$ is an equivalence relation

(2) composition in $S^{-1}\mathcal{C}$ is well-defined

(3) $S^{-1}\mathcal{C}$ is a category (except perhaps that the Hom-spaces may not be se

(4) Have <u>isomorphism</u> of categories $F: S^{-1}\mathcal{C} \xrightarrow{\cong} \mathcal{C}[S^{-1}]$ given by identity on objects, and sending right fraction $X \xleftarrow{s} Y' \xrightarrow{\alpha} Y$ to $[X, \alpha, s^{-1}, Y]$

Back to complexes: $\mathcal{A}$ abelian.

Want to localize at quasi-iso's, i.e. morphisms $X^\bullet \xrightarrow{f^\bullet} Y^\bullet$ s.t. $H^n(f^\bullet)$ iso $\forall n \in \mathbb{Z}$.

<u>Problem</u>: $\{$quasi-iso's$\} \subset \mathrm{Mor}\, Ch(\mathcal{A})$ <u>does not</u> admit a calculus of right (or left) fractions.

<u>Solution</u>: Consider $K(\mathcal{A})$ instead of $Ch(\mathcal{A})$

<u>Proposition</u>: $S = \{[f] \mid f$ quasi-iso in $Ch(\mathcal{A})\} \subset \mathrm{Mor}\, K(\mathcal{A})$ admits a calculus of right and left fractions.

<u>Pf</u>: (RF1) clear

(RF2) & $(RF2)^{op}$ - see Lemma 34.3 in notes (Ore condition)

(RF3) since $K(\mathcal{A})$ additive, suffices to show:

$X \xrightarrow{h} Y \xrightarrow{s} Y'$ $\quad s \circ h = 0$, s quasi-iso

$\Rightarrow \exists$ quasi-iso $t: X' \to X$ s.t. $h \circ t = 0$

Assume h is as above

Apply $\mathrm{Hom}_{K(\mathcal{A})}(X,-)$ to triangle

$Y \xrightarrow{s} Y' \xrightarrow{k} \mathrm{Cone}(s) \xrightarrow{e} Y[1]$ gives l.e.s.

$(X, \mathrm{Cone}(s)[-1]) \xrightarrow{e[-1]\circ -} (X, Y) \xrightarrow{s \circ -} (X, Y') \xrightarrow{k \circ -} (X, \mathrm{Cone}(s)) \to \dots$

$s \circ h = 0 \Rightarrow \exists\ h' : X \to \mathrm{Cone}(s)[-1]$ s.t.

$$\begin{tikzcd} X \arrow[d, "h'"'] \arrow[dr, "h"] & \\ \mathrm{Cone}(s)[-1] \arrow[r, "e[-1]"'] & Y \end{tikzcd}$$

commutes.

Set $\mathrm{Cone}(h')[-1] = X'$, get triangle $X' \xrightarrow{t} X \xrightarrow{h'} \mathrm{Cone}(s)[-1] \to X'[1]$

$X' \xrightarrow{t} X$ quasi-iso since $\mathrm{Cone}(t) \cong \mathrm{Cone}(s)[-1]$ in $K(\mathcal{A})$, and $\mathrm{Cone}(s)[-1]$ is exact.

Finally, h factors through $\mathrm{Cone}(s)[-1]$ $\Rightarrow h \circ t = 0$. This shows (RF3). $(RF3)^{op}$ shown dually ■

<u>Definition</u>: $\mathcal{A}$ - abelian cat.

The <u>derived category of $\mathcal{A}$</u>, denoted $D(\mathcal{A})$, is the localization of $K(\mathcal{A})$ w.r.t class of quasi-isomorphisms, $S^{-1}K(\mathcal{A})$

<u>Remarks</u> · $\mathrm{Ob}\, D(A) = \mathrm{Ob}\, K(\mathcal{A}) = \mathrm{Ob}\, Ch(\mathcal{A})$
chain complexes

Mor $D(\mathcal{A})$ - right fractions of morphisms in $K(\mathcal{A})$.

- $D(\mathcal{A})$ may not be a cat, if Hom-spaces are not sets. Need

$$\mathrm{Hom}_{D(\mathcal{A})}(X^\bullet, Y^\bullet) = \{\text{right fractions from } X^\bullet \text{ to } Y^\bullet\}$$

to be a set $\forall X^\bullet, Y^\bullet \in D(\mathcal{A})$. This holds in many cases, e.g. if $\mathcal{A} = \mathrm{Mod}\,R$, R ring. We always assume this holds.

What properties does $D(\mathcal{A})$ have?

<u>Lemma</u>: $\mathcal{B}$ additive cat, $S \subset \mathrm{Mor}\,\mathcal{B}$ admits calculus of right fractions. Then

(i) $S^{-1}\mathcal{B}$ is additive

(ii) The functor $\mathcal{B} \xrightarrow{q} S^{-1}\mathcal{B}$ is additive

<u>Pf</u> Define addition as follows:

Given $\left(X \xleftarrow{s_1 \in S} X_1 \xrightarrow{\alpha} Y\right) \in \mathrm{Mor}\,S^{-1}\mathcal{B}$

$$\left(X \xleftarrow{s_2 \in S} X_2 \xrightarrow{\beta} Y\right) \in \mathrm{Mor}\,S^{-1}\mathcal{B}.$$

want to add $[s_1, \alpha]$ and $[s_2, \beta]$.

By (RF2), can complete diagram

$$\begin{tikzcd} X' \arrow[r, dashed, "s_1' \in S"] \arrow[d, dashed, "s_2'"'] & X_2 \arrow[d, "s_2"] \\ X_1 \arrow[r, "s_1"] & X \end{tikzcd}$$

Set $s = s_2 s_1' = s_1 s_2'$

S closed under composition $\Rightarrow s \in S$

Have comm diagrams

$$\begin{tikzcd} & X_1 \arrow[dl, "s_1"'] \arrow[dr, "\alpha"] & \\ X & & Y \\ & X' \arrow[ul, "s"] \arrow[uu, "s_2'"] \arrow[ur, "\alpha\circ s_2'"'] & \end{tikzcd} \qquad \begin{tikzcd} & X_2 \arrow[dl, "s_2"'] \arrow[dr, "\beta"] & \\ X & & Y \\ & X' \arrow[ul, "s"] \arrow[uu, "s_1'"] \arrow[ur, "\beta\circ s_1'"'] & \end{tikzcd}$$

$\Rightarrow [s_1, \alpha] = [s, \alpha\circ s_2'] \;\&\; [s_2, \beta] = [s, \beta\circ s_1']$

"common denominator"

$$[s_1, \alpha] + [s_2, \beta] = [s, \alpha\circ s_2'] + [s, \beta\circ s_1']$$
$$= [s, \alpha\circ s_2' + \beta\circ s_1'].$$

<u>Check</u>: This is well-defined

<u>Check</u>: This makes $\mathrm{Hom}_{S^{-1}\mathcal{B}}(X, Y)$ into an abelian group, with neutral element $(X \xleftarrow{1} X \xrightarrow{0} Y)$

<u>Check</u>: composition in $S^{-1}\mathcal{B}$ is bilinear
$\Rightarrow S^{-1}\mathcal{B}$ is preadditive.

Clearly, with this definition, the functor $\mathcal{B} \xrightarrow{q} S^{-1}\mathcal{B}$ is additive.

remains to show: $S^{-1}\mathcal{B}$ - has a zero obj
- has biproducts