

- the third column is isomorphic in $K(\mathcal{A})$ to a standard triangle.

This is left as an exercise.

Next, want to show $\text{Ext}_{\mathcal{A}}^n(A, B)$ iso to morphism spaces in $K(\mathcal{A})$.

Lemma: $X \in \mathcal{A}$ $A = (\cdots \rightarrow A^1 \rightarrow A^0 \rightarrow A^{-1} \rightarrow \cdots)$. (consider

$$\text{Hom}_{\mathcal{A}}(A, X) = (\cdots \rightarrow \text{Hom}_{\mathcal{A}}(A^1, X) \rightarrow \text{Hom}_{\mathcal{A}}(A^0, X) \rightarrow \text{Hom}_{\mathcal{A}}(A^{-1}, X) \rightarrow \cdots)$$

Then

$$Z^n \text{Hom}_{\mathcal{A}}(A, X) = \text{Hom}_{\text{Ch}(\mathcal{A})}(A, X[n])$$

$$H^n \text{Hom}_{\mathcal{A}}(A, X) = \text{Hom}_{K(\mathcal{A})}(A, X[n])$$

Pf: Exercise.

$p: \mathcal{A} \rightarrow K(\mathcal{A})$ $X \mapsto pX$ - proj res of X
 Have functor $\mathcal{A} \rightarrow K(\mathcal{A})$ $A \mapsto (\cdots \rightarrow 0 \rightarrow A \rightarrow 0 \rightarrow \cdots)$
 complex concentrated in degree 0

Theorem: \mathcal{A} abelian with enough proj. Then

$$\text{Ext}_{\mathcal{A}}^n(A, B) = \text{Hom}_{K(\mathcal{A})}(pA, B[n]).$$

Pf: $\text{Ext}_{\mathcal{A}}^n(A, B) \stackrel{\text{def}}{=} R\text{Hom}(-B)(A)$

$$\stackrel{\text{def}}{=} H^n \text{Hom}_{\mathcal{A}}(pA, B[n]) \stackrel{\text{Lemma}}{\cong} \text{Hom}_{K(\mathcal{A})}(pA, B[n]).$$

• Dual versions of two previous results also hold.

• Let $0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0$ \neq s.e.s in \mathcal{A}

By horseshoe lemma, can choose proj resolutions

pA_1, pA_2, pA_3 s.t. $0 \rightarrow pA_1 \rightarrow pA_2 \rightarrow pA_3 \rightarrow 0$

componentwise split exact sequence of complexes,

\Rightarrow gives a triangle $pA_1 \rightarrow pA_2 \rightarrow pA_3 \rightarrow pA_1[1] \in \Delta$
in $K(\mathcal{A})$

Applying $\text{Hom}(-, B)$ & using $K(\mathcal{A})$ triangulated
get exact seq

$$\cdots \rightarrow (pA_3[n], B) \rightarrow (pA_2[n], B) \rightarrow (pA_1[n], B) \rightarrow (pA_3[n-1], B) \rightarrow \cdots$$

Using previous theorem get long exact sequence
in Hom and Ext . This is the same as long
exact seq coming from applying $\text{Hom}(-, B)$ to $(*)$
and considering long exact seq from its derived
functors.

Derived categories

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- Want to identify complexes up to cohomology
i.e. up to quasi-isomorphism.

Idea: Construct new category where we "force" quasi-iso's to be isomorphisms.

Def: \mathcal{L} category, S class of morphisms in \mathcal{L}

The localisation of \mathcal{L} w.r.t S is a pair $(\mathcal{L}[S^{-1}], Q)$ where

- $\mathcal{L}[S^{-1}]$ is a category

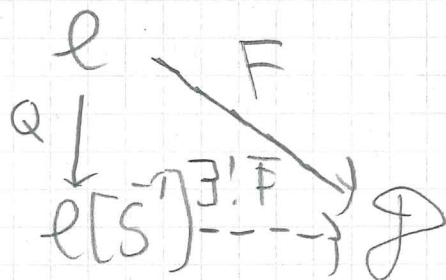
- $Q: \mathcal{L} \rightarrow \mathcal{L}[S^{-1}]$ is a functor

s.t.

(1) $s \in S \Rightarrow Q(s)$ is an iso

(2) $F: \mathcal{L} \rightarrow \mathcal{D}$ functor s.t. $F(s)$ iso $\forall s \in S$

$\Rightarrow \exists$ unique functor $\bar{F}: \mathcal{L}[S^{-1}] \rightarrow \mathcal{D}$ s.t. $F = \bar{F} \circ Q$



Note: $\mathcal{L}[S^{-1}]$ unique up to unique iso, since it is defined by a universal property
Does it exist?

Naively: $Ob \ell[S^{-1}] = Ob \ell$

$$S^{-1} = \{f^{-1} : Y \rightarrow X \mid f : X \rightarrow Y \in S \text{ Mor } \ell\}$$

$$\text{Mor } \ell[S^{-1}](X, Y) = \left\{ \text{strings } (X, p_1, \dots, p_n, Y) \mid p_i \in \text{Mor } \ell \cup S^{-1} \right\}$$

codomain $p_i = \text{domain } p_{i+1}$
domain $p_1 = X$ codomain $p_n = Y$

where \sim - smallest equivalence relation s.t.

(1) If $(X, \sigma_1, Y) \sim (X, \sigma_2, Y)$, then

σ_1, σ_2 strings

- $(X, \sigma \sigma_1, Z) \sim (X, \sigma \sigma_2, Z)$ for all strings (X, σ, Z)
- $(W, \sigma_1 \sigma, Y) \sim (W, \sigma_2 \sigma, Y)$ for all strings (W, σ, X)
- $(X, id_X, X) \sim (X, \emptyset, X)$
- $(X, f, g, Y) \sim (X, f \circ g, Y)$ if $f, g \in \text{Mor } \ell$
- $(X, f, f^{-1}, X) \sim (X, id_X, X) \sim (X, f^{-1}, f, X)$ for $f \in S$

- composition = concatenation of strings.

$$\underline{\text{Ex:}} \quad X_1 \xrightarrow{f_1} X_2 \xleftarrow{f_2} X_3 \xrightarrow{f_3} X_4 \xrightarrow{f_4} X_5 \xleftarrow{f_4} X_4$$

with $f_2, f_4 \in S$ gives morphism

$$(X_1, \overline{f_4}, \overline{f_4} \circ f_3, \overline{f_2}, \overline{f_1}, X_4)$$

$$\sim (X_1, \underbrace{\overline{f_4}}_{\sim \text{id}_{X_1}}, \overline{f_4}, f_3, \overline{f_2}, \overline{f_1}, X_4)$$

$$\sim (X_1, f_3, \overline{f_2}, \overline{f_1}, X_4)$$

Have functor $\mathcal{Q}^l \rightarrow \mathcal{L}[S^{-1}]$

$$(X \xrightarrow{f} Y) \mapsto (X, f, Y)$$

Check: $(\mathcal{L}[S^{-1}], \mathcal{Q})$ satisfies universal property of localization

Problem: $\text{Mor } \mathcal{L}[S^{-1}]$ can be very complicated

$\text{Hom}_{\mathcal{L}[S^{-1}]}(X, Y)$ may not be a set!

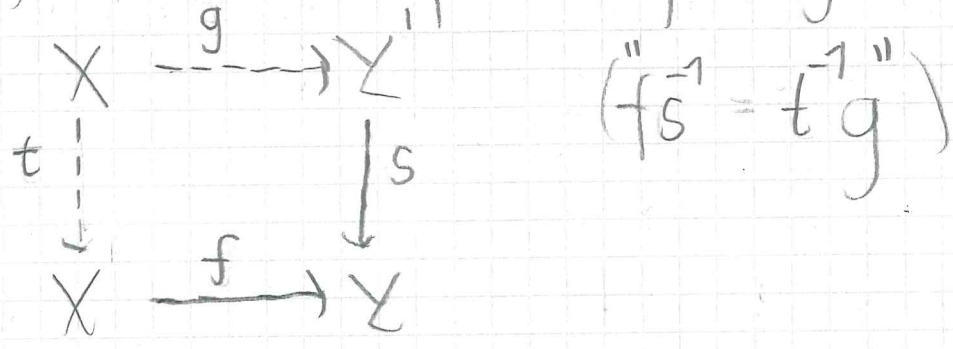
$\Rightarrow \mathcal{L}[S^{-1}]$ may not be a category

\rightarrow Identify properties of S which makes $\mathcal{L}[S^{-1}]$ "well-behaved"

Def. $S \subset \text{Mor } \mathcal{L}$ admits a calculus of right fractions if the following hold:

(RF1) $\text{id}_X \in S \quad \forall X \in \mathcal{L}$, and S is closed under composition

(RF2) Given solid part of diagram



with $s \in S$, can find dashed arrow s.t. square commutes and $t \in S$

(RF3) $\alpha, \beta: X \rightarrow Y$ morphisms in \mathcal{L}
 If $\exists s \in S$ s.t. $ts\alpha = s\beta$ then $\exists t \in S$
 s.t. $\alpha t = \beta t$

S admits calculus of left fractions if \mathcal{L}^{op} admits calculus of right fractions in \mathcal{L}^{op} .

Assume \mathcal{C} admits calculus of right fractions,

pair $X \xleftarrow{s} Y \xrightarrow{\alpha} Z$ with $s \in S$

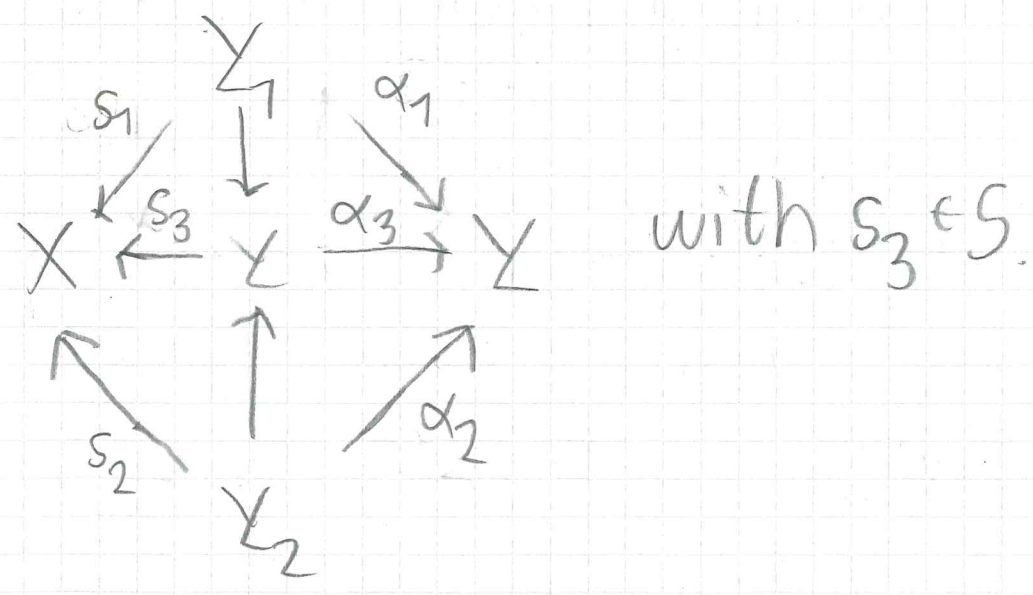
called right fraction,

Define cat $S^{-1}\mathcal{C}$:

$Ob S^{-1}\mathcal{C} = Ob \mathcal{C}$

$Mor_{S^{-1}\mathcal{C}}(X, Y) =$ equivalence classes of right fractions $X \xleftarrow{s} Y' \xrightarrow{\alpha} Y$, where

$(s_1, \alpha_1) \sim (s_2, \alpha_2)$ if \exists comm diagram



(Composite of $[s, \alpha]$ and $[t, \beta]$) is $[s \circ t, \beta \circ \alpha]$ where