- the third column is isomorphic in $K(\mathcal{A})$ to a standard triangle.

This is left as an exercise ∎

Next, want to show $\mathrm{Ext}^n_{\mathcal{A}}(A,B)$ iso to morphism spaces in $K(\mathcal{A})$

<u>Lemma</u>: $X \in \mathcal{A}$ $A^\bullet = (\cdots \to A^{-1} \to A^0 \to A^1 \to \cdots)$. Consider

$$\mathrm{Hom}_{\mathcal{A}}(A^\bullet, X) = (\cdots \to \mathrm{Hom}_{\mathcal{A}}(A^1, X) \to \mathrm{Hom}_{\mathcal{A}}(A^0, X) \to \mathrm{Hom}_{\mathcal{A}}(A^{-1}, X) \to \cdots$$

Then

$$Z^n \mathrm{Hom}_{\mathcal{A}}(A^\bullet, X) = \mathrm{Hom}_{Ch(\mathcal{A})}(A^\bullet, X[n])$$
$$H^n \mathrm{Hom}_{\mathcal{A}}(A^\bullet, X) = \mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, X[n])$$

<u>Pf</u>: Exercise ∎

$p : \mathcal{A} \to K(\mathcal{A})$ $X \longmapsto pX$ - proj res of X

Have functor $\mathcal{A} \to K(\mathcal{A})$, $A \longmapsto (\cdots \to 0 \to A \to 0 \to \cdots)$
complex concentrated in degree 0

<u>Theorem</u>: $\mathcal{A}$ abelian with enough proj. Then
$$\mathrm{Ext}^n_{\mathcal{A}}(A,B) = \mathrm{Hom}_{K(\mathcal{A})}(pA, B[n])$$

<u>Pf</u>: $\mathrm{Ext}^n_{\mathcal{A}}(A,B) \overset{def}{=} R^n\mathrm{Hom}(-,B)(A)$
$\overset{def}{=} H^n \mathrm{Hom}_{\mathcal{A}}(pA, B[n]) \overset{\text{Lemma}}{=} \mathrm{Hom}_{K(\mathcal{A})}(pA, B[n])$ ∎

- Dual versions of two previous results also hold.

- Let $0 \to A_1 \to A_2 \to A_3 \to 0$ (*) s.e.s in $\mathcal{A}$

By horseshoe lemma, can choose proj resolutions pA_1, pA_2, pA_3 s.t $0 \to pA_1 \to pA_2 \to pA_3 \to 0$ componentwise split exact sequence of complexes.

$\Rightarrow$ gives a triangle $pA_1 \to pA_2 \to pA_3 \to pA_1[1] \in \Delta$ in $K(\mathcal{A})$

Applying $\mathrm{Hom}(-, B)$ & using $K(\mathcal{A})$ triangulated get exact seq

$$\cdots \to (pA_3[n], B) \to (pA_2[n], B) \to (pA_1[n], B) \to (pA_3[n-1], B) \to \cdots$$

Using previous theorem get long exact sequence in Hom and Ext. This is the same as long exact seq coming from applying $\mathrm{Hom}(-, B)$ to (*) and considering long exact seq from its derived functors.

Derived categories

- Want to identify complexes up to cohomology i.e. up to quasi-isomorphism.

Idea: Construct new category where we "force" quasi-iso's to be isomorphisms.

<u>Def</u>: $\mathcal{C}$ category, S class of morphisms in $\mathcal{C}$

The localisation of $\mathcal{C}$ w.r.t S is a pair $(\mathcal{C}[S^{-1}], Q)$ where

- $\mathcal{C}[S^{-1}]$ is a category
- $Q: \mathcal{C} \longrightarrow \mathcal{C}[S^{-1}]$ is a functor

s.t.

(1) $s \in S \Rightarrow Q(s)$ is an iso

(2) $F: \mathcal{C} \to \mathcal{D}$ functor s.t $F(s)$ iso $\forall s \in S$

$\Rightarrow \exists$ <u>unique</u> functor $\bar{F}: \mathcal{C}[S^{-1}] \to \mathcal{D}$ s.t. $F = \bar{F} \circ Q$

$$\begin{tikzcd} \mathcal{C} \arrow[d, "Q"'] \arrow[dr, "F"] & \\ \mathcal{C}[S^{-1}] \arrow[r, dashed, "\exists ! \bar{F}"] & \mathcal{D} \end{tikzcd}$$

<u>Note</u>: $\mathcal{C}[S^{-1}]$ unique up to unique iso, since it is defined by a universal property

Does it exist?

<u>Naively</u>: $\mathrm{Ob}\,\mathcal{C}[S^{-1}] = \mathrm{Ob}\,\mathcal{C}$

$\bar{S} = \{\bar{f} : Y \to X \mid f : X \to Y \in S \subset \mathrm{Mor}\,\mathcal{C}\}$

$$\mathrm{Mor}\,\mathcal{C}[S^{-1}](X,Y) = \Big\{\text{strings } (X, \overbrace{p_n, \dots, p_1}^{\sigma}, Y) \mid p_i \in \mathrm{Mor}\,\mathcal{C} \cup \bar{S},\ \text{codomain } p_i = \text{domain } p_{i+1},\ \text{domain } p_1 = X,\ \text{codomain } p_n = Y\Big\} \Big/ \sim$$

where $\sim$ - smallest equivalence relation s.t.

(1) If $(X, \sigma_1, Y) \sim (X, \sigma_2, Y)$, then

σ_1, σ_2 strings

- $(X, \sigma\sigma_1, Z) \sim (X, \sigma\sigma_2, Z)$ for all strings (Y, σ, Z)
- $(W, \sigma_1\sigma, Y) \sim (W, \sigma_2\sigma, Y)$ for all strings (W, σ, X)
- $(X, id_X, X) \sim (X, \phi, X)$
- $(X, f, g, Y) \sim (X, f \circ g, Y)$ if $f, g \in \mathrm{Mor}\,\mathcal{C}$
- $(X, f, \bar{f}, X) \sim (X, id_X, X) \sim (X, \bar{f}, f, X)$ for $f \in S$

- composition = concatenation of strings.

Ex: $X_1 \xrightarrow{f_1} X_2 \xleftarrow{f_2} X_3 \xrightarrow{f_3} X_4 \xrightarrow{f_4} X_5 \xleftarrow{f_4} X_4$

with $f_2, f_4 \in S$ gives morphism

$(X_1, \bar{f_4}, f_4 \circ f_3, \bar{f_2}, f_1, X_4)$

$\sim (X_1, \underbrace{\bar{f_4}, f_4}_{\sim id_{X_1}}, f_3, \bar{f_2}, f_1, X_4)$

$\sim (X_1, f_3, \bar{f_2}, f_1, X_4)$.

Have functor $Q: \mathcal{C} \to \mathcal{C}[S^{-1}]$

$(X \xrightarrow{f} Y) \mapsto (X, f, Y)$

Check: $(\mathcal{C}[S^{-1}], Q)$ satisfies universal property of localization

Problem: $\operatorname{Mor} \mathcal{C}[S^{-1}]$ can be very complicated

$\operatorname{Hom}_{\mathcal{C}[S^{-1}]}(X, Y)$ may not be a set!

$\Rightarrow \mathcal{C}[S^{-1}]$ may not be a category

$\to$ Identify properties of S which makes $\mathcal{C}[S^{-1}]$ "well-behaved"

<u>Def</u>: $S \subset \mathrm{Mor}\,\mathcal{C}$ admits a <u>calculus of right fractions</u> if the following hold:

(RF1) $id_X \in S$ $\forall X \in \mathcal{C}$, and S is closed under composition

(RF2) Given solid part of diagram,

$$\begin{tikzcd} X \arrow[r, dashed, "g"] \arrow[d, dashed, "t"'] & Y \arrow[d, "s"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \quad (\text{"} f s^{-1} = t^{-1} g \text{"})$$

with $s \in S$, can find dashed arrow s.t. square commutes and $t \in S$

(RF3) $\alpha, \beta : X \to Y$ morphisms in $\mathcal{C}$
If $\exists s \in S$ s.t. $s\alpha = s\beta$ then $\exists t \in S$ s.t. $\alpha t = \beta t$

S admits <u>calculus of left fractions</u> if admits calculus of right fractions in $\mathcal{C}^{op}$.

Assume $\mathcal{C}$ admits calculus of right fractions.

pair $X \xleftarrow{s} Y' \xrightarrow{\alpha} Y$ with $s \in S$

called <u>right fraction</u>,

Define cat $S^{-1}\mathcal{C}$:

$\mathrm{Ob}\, S^{-1}\mathcal{C} = \mathrm{Ob}\,\mathcal{C}$

$\mathrm{Mor}_{S^{-1}\mathcal{C}}(X, Y)$ = equivalence classes of right fractions $X \xleftarrow{s} Y' \xrightarrow{\alpha} Y$, where

$(s_1, \alpha_1) \sim (s_2, \alpha_2)$ if $\exists$ comm diagram

$$\begin{tikzcd}
 & Y_1 \arrow[dl, "s_1"'] \arrow[d] \arrow[dr, "\alpha_1"] & \\
X & Y \arrow[l, "s_3"'] \arrow[r, "\alpha_3"] & Y \\
 & Y_2 \arrow[ul, "s_2"] \arrow[u] \arrow[ur, "\alpha_2"'] &
\end{tikzcd}$$

with $s_3 \in S$.

Composite of $[s, \alpha]$ and $[t, \beta]$ is $[s \circ t', \beta \circ \alpha']$ where