

Example (Free modules) Ring

Forgetful functor $F: \text{Mod } R \rightarrow \text{Set}$
 has a left-adjoint $R^{(-)}: \text{Set} \rightarrow \text{Mod } R$
 given as follows

$$X \xrightarrow{\cong} R^{(X)} = \left\{ \text{maps } f: X \rightarrow R \mid \begin{array}{l} f(x) \neq 0 \text{ for only} \\ \text{finitely many } x \in X \end{array} \right\}$$

\cong
set

$\varphi: X \rightarrow Y$ maps in Set. Then

$$R^{(\varphi)}: R^{(X)} \rightarrow R^{(Y)} \text{ given by}$$

$$R^{(\varphi)}(f): Y \rightarrow R$$

$$y \longmapsto \sum_{x \in \varphi^{-1}(y)} f(x)$$

The adjunction iso is

$$\text{Hom}_R(R^{(X)}, M) \xrightarrow{\cong} \text{Hom}_{\text{Set}}(X, F(M))$$

$$\psi \longmapsto [x \mapsto \psi(\chi_x)]$$

$$[f \mapsto \sum_{x \in X} f(x) \varphi(x)] \longleftarrow \varphi$$

where $\chi_x: X \rightarrow R$, $\chi_x(y) = \begin{cases} 0 & x \neq y \\ 1_R & x = y \end{cases}$

Proposition (Unit-counit adjunction)

$F: \mathcal{C} \rightarrow \mathcal{D}$ $G: \mathcal{D} \rightarrow \mathcal{C}$ functors.

The following are equivalent:

1. (F, G) is an adjoint pair.

2. \exists natural transformations

$$\eta: \text{Id}_{\mathcal{C}} \rightarrow G \circ F \quad \& \quad \varepsilon: F \circ G \rightarrow \text{Id}_{\mathcal{D}}$$

s.t. $\forall X \in \mathcal{C}, Y \in \mathcal{D}$ the diagrams

$$\begin{array}{ccc} F(X) & \xrightarrow{F(\eta_X)} & FG(X) \\ \searrow \text{id}_{F(X)} & & \downarrow \varepsilon_{F(X)} \\ & & F(X) \end{array} \quad \begin{array}{ccc} G(Y) & \xrightarrow{\eta_{G(Y)}} & FG(Y) \\ \searrow \text{id}_{G(Y)} & & \downarrow G(\varepsilon_Y) \\ & & G(Y) \end{array}$$

commutes.

Sketch of proof:

Let $\phi_{X,Y}: \text{Hom}_{\mathcal{D}}(F(X), Y) \rightarrow \text{Hom}_{\mathcal{C}}(X, G(Y))$

adjunction iso. Define

$$\eta_X = \phi_{X, FG(X)}(\text{id}_{F(X)}): X \rightarrow G(F(X))$$

$$\varepsilon_Y = \phi_{FG(Y), Y}^{-1}(\text{id}_{G(Y)}): F(G(Y)) \rightarrow Y.$$

Check:

$$\eta = \{\eta_x \mid X \in \mathcal{C}\} \text{ \& \ } \varepsilon = \{\varepsilon_Y \mid Y \in \mathcal{D}\}$$

are natural transformations

$$\varepsilon_{F(X)} \circ F(\eta_x) = \text{id}_{F(X)} \text{ \& \ } G(\varepsilon_Y) \circ \eta_{G(Y)} = \text{id}_{G(Y)}$$

Conversely, let $\eta: \text{id}_{\mathcal{C}} \rightarrow G \circ F$ and $\varepsilon: F \circ G \rightarrow \text{id}_{\mathcal{D}}$ be as in (2).

Define

$$\phi_{X,Y}: \text{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{G(\cdot)} \text{Hom}_{\mathcal{C}}(GF(X), G(Y))$$

$\xrightarrow{\text{Hom}_{\mathcal{C}}(\eta_x, G(\cdot))} \text{Hom}_{\mathcal{C}}(X, G(Y))$

Check: $\phi_{X,Y}$ is an iso $\forall X \in \mathcal{C}, Y \in \mathcal{D}$

$\phi = \{\phi_{X,Y}\}$ is natural.

Limits

Def: \mathcal{I} small cat ($\text{Ob } \mathcal{I}$ is a set),
 \mathcal{C} cat, $D: \mathcal{I} \rightarrow \mathcal{C}$ functor.

• A cone (C, q_i) of D is an object $C \in \mathcal{C}$
together with morphisms

$$\{q_i: C \rightarrow D(i) \mid i \in \mathcal{I}\} \text{ s.t.}$$

$$\begin{array}{ccc} & q_i & \rightarrow D(i) \\ C & \searrow & \downarrow D(f) \\ & q_j & \rightarrow D(j) \end{array}$$

commutes \forall morphisms
 f in \mathcal{I}

• A limit (C, p_i) of D is a cone such that
for any other cone (C', p'_i) of D
there exists a unique morphism

$$\varphi: C' \rightarrow C \text{ s.t.}$$

$$\begin{array}{ccc} C' & \xrightarrow{\varphi} & C \\ p'_i \downarrow & & \downarrow p_i \\ & & D(i) \end{array}$$

commutes $\forall i \in \text{Ob } \mathcal{I}$.

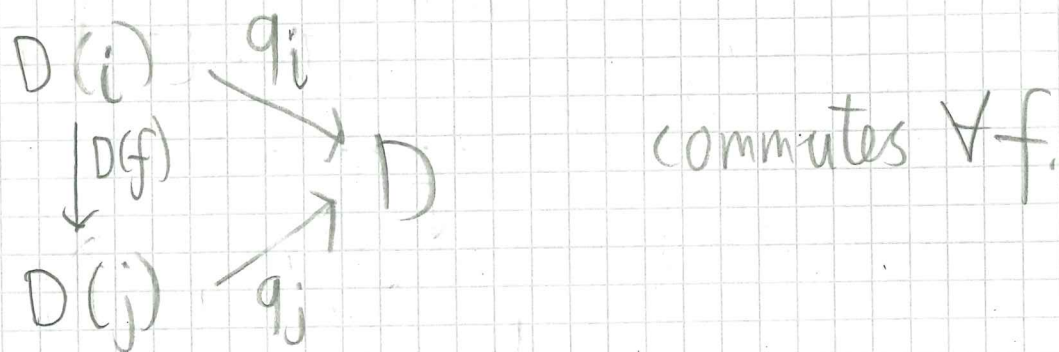
write $C = \varprojlim D$

Dual definitions using e° :

• A cocone (C, q_i) of D is

• $C \in \mathcal{C}$

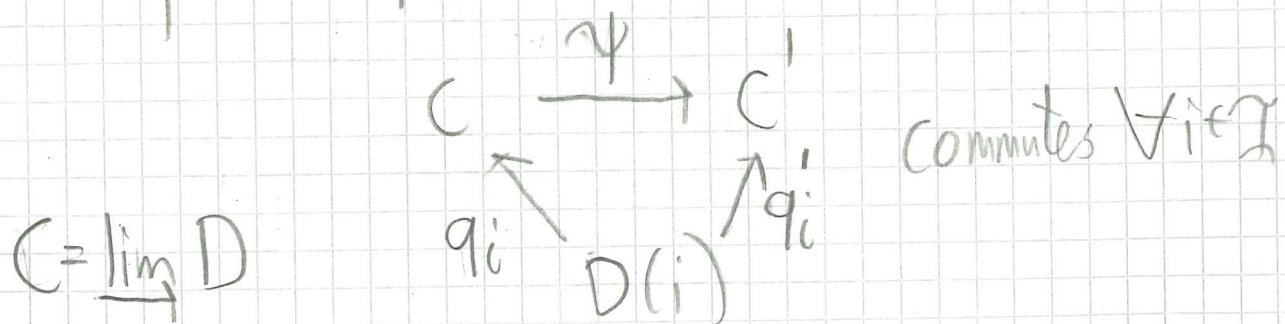
• $q_i: D(i) \rightarrow C$ s.t.



• A colimit of D is a cocone (C, q_i)

s.t. \forall cocones $(C', q'_i) \exists$ unique

morphism $\psi: C \rightarrow C'$ s.t.



Remark:

• In the notes limits/colimits are defined for contravariant functors $\mathcal{I}^{\circ} \rightarrow \mathcal{C}$.

• limits & colimits are unique if they exist (check!)

• Ex(product) X set, $\{C_x \mid x \in X\}$ set of objects in \mathcal{C} .

Define cat \mathcal{E}_X .

$$\text{Obj } \mathcal{E}_X = X$$

$$\text{Hom}_{\mathcal{E}_X}(x, y) = \begin{cases} \emptyset & x \neq y \\ \{\text{id}_x\} & x = y \end{cases}$$

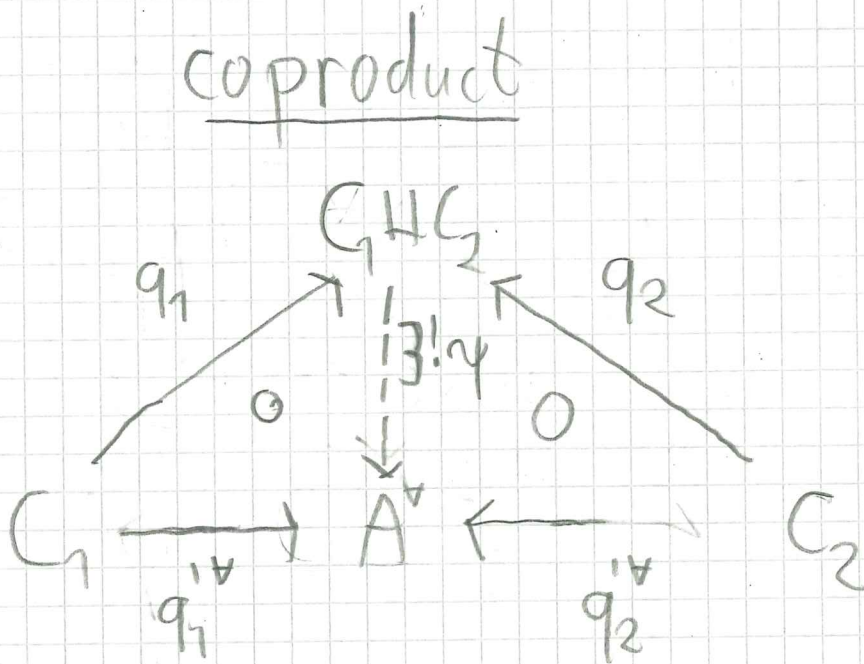
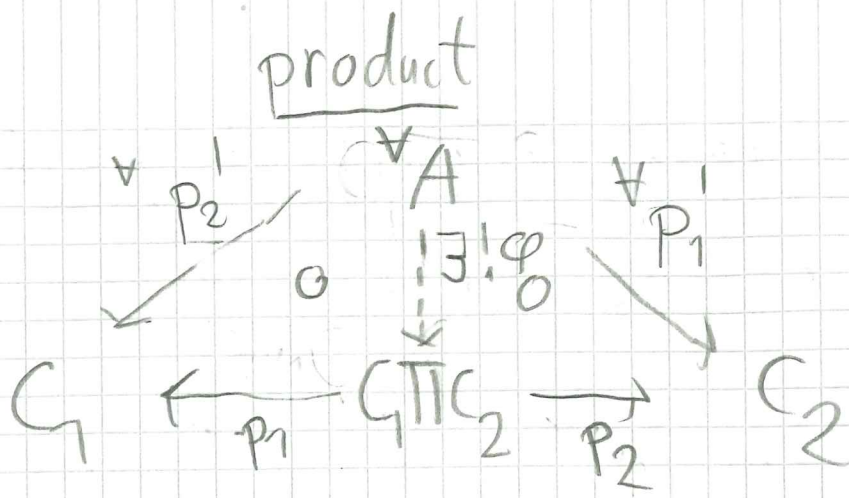
$x \mapsto C_x$ defines a functor

$$D: \mathcal{E}_X \longrightarrow \mathcal{C}$$

• If $\varprojlim D$ exists, it is called the product of $\{C_x\}$, denoted $\prod_{x \in X} C_x$

• If $\varinjlim D$ exists, it is called the coproduct of $\{C_x\}$, denoted $\coprod_{x \in X} C_x$

For example, for $C, C' \in \mathcal{C}$:

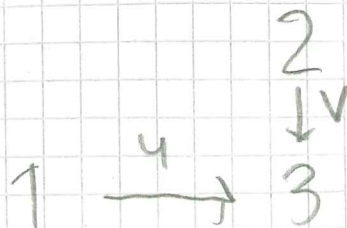


For Set
 products = cartesian products
 coproducts = disjoint unions. (check!)

Example (pullbacks & pushouts)

(1) let \mathcal{A} be the cat

e. $\text{Obj } \mathcal{A} = \{1, 2, 3\}$



$$\text{Hom}_{\mathcal{I}}(1, 2) = \{u\} \quad \text{Hom}_{\mathcal{I}}(1, 3) = \emptyset$$

$$\text{Hom}_{\mathcal{I}}(1, 1) = \{\text{id}_1\} \dots$$

A functor $D: \mathcal{I} \rightarrow \mathcal{C}$ is given by a diagram

$$\begin{array}{ccc}
 & & C_2 \\
 & & \downarrow \beta \\
 C_1 & \xrightarrow{\alpha} & C_3
 \end{array}
 \quad \left(\begin{array}{l} D(i) = C_i \\ D(u) = \alpha \\ D(v) = \beta \end{array} \right)$$

The limit of D is called a pullback of the diagram. Given by

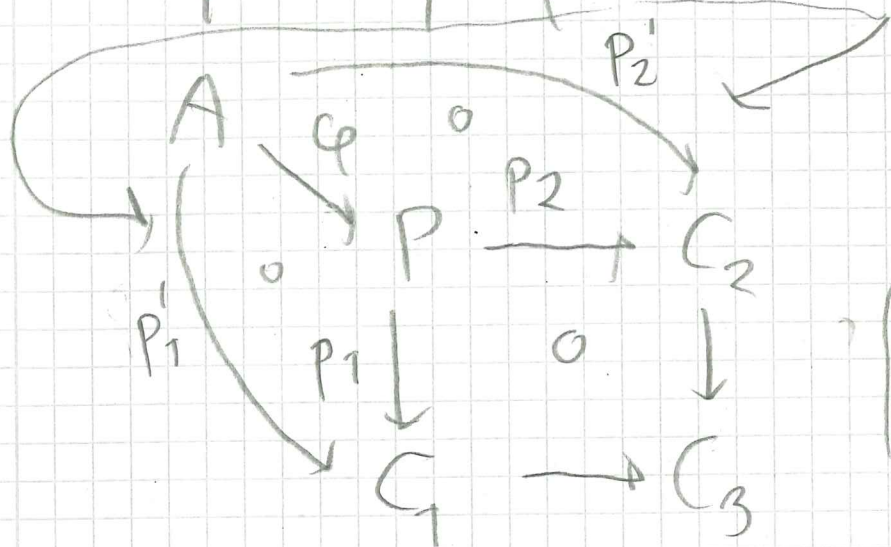
commutative diagram
(sometimes called cartesian)

$$\begin{array}{ccc}
 P & \xrightarrow{p_2} & C_2 \\
 p_1 \downarrow & & \downarrow \beta \\
 C_1 & \xrightarrow{\alpha} & C_3
 \end{array}$$

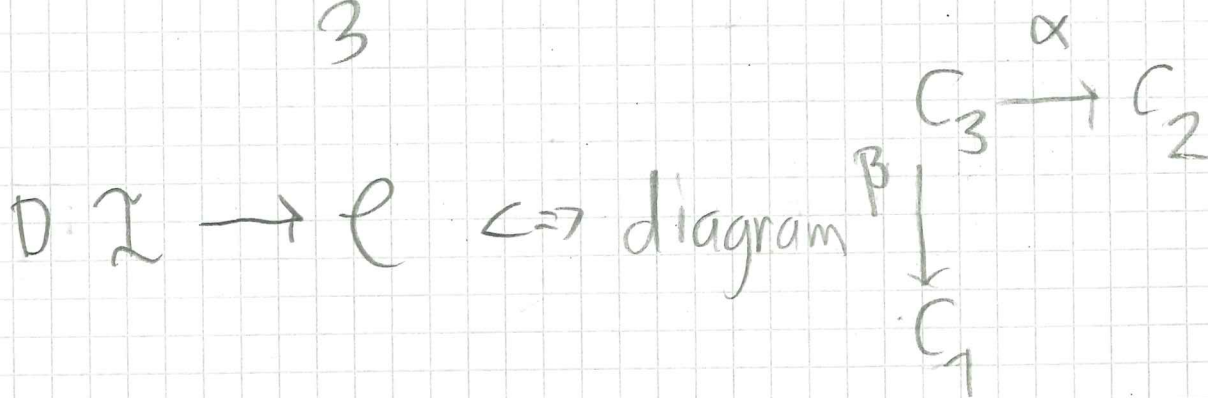
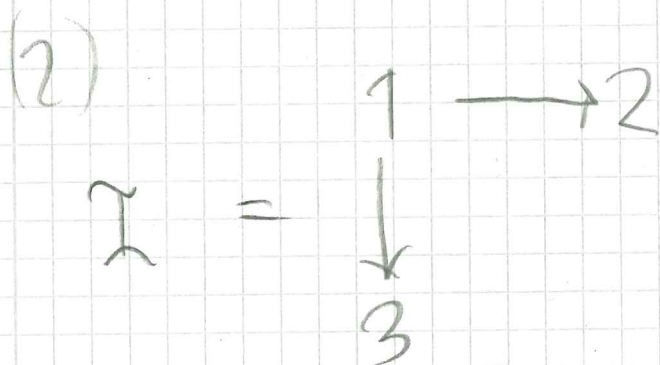
s.t. $\forall A \in \mathcal{C}$ & commutative diagrams

$$\begin{array}{ccc}
 A & \xrightarrow{p_2} & C_2 \\
 p_1 \downarrow & & \downarrow \beta \\
 C_1 & \xrightarrow{\alpha} & C_3
 \end{array}$$

\exists unique map $\varphi: A \rightarrow P$ s.t commutes.

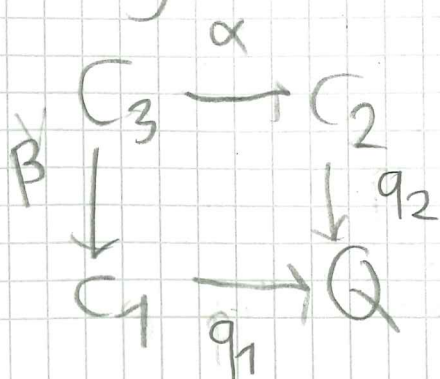


(what happens to p_3 & q_3 ?)

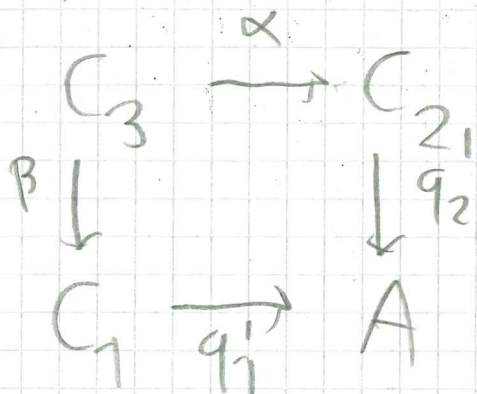


A colimit of F is called a pushout

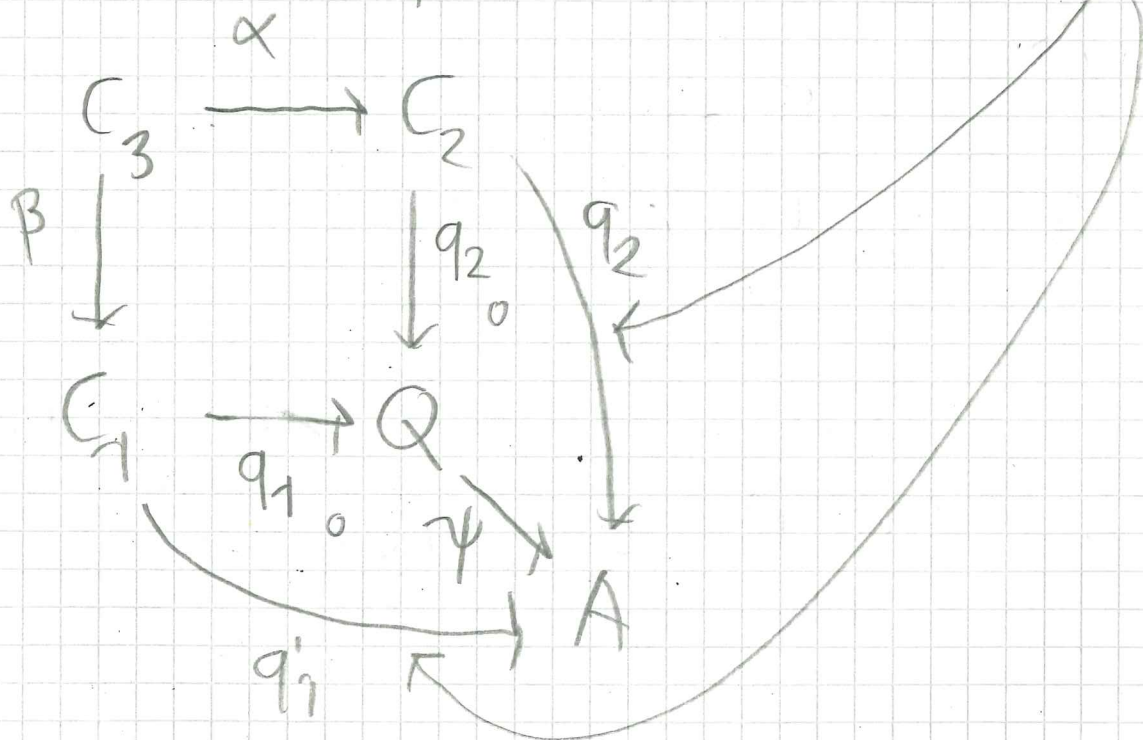
Given by comm diagram (sometimes called cocartesian)



s.t. for all comm diagram



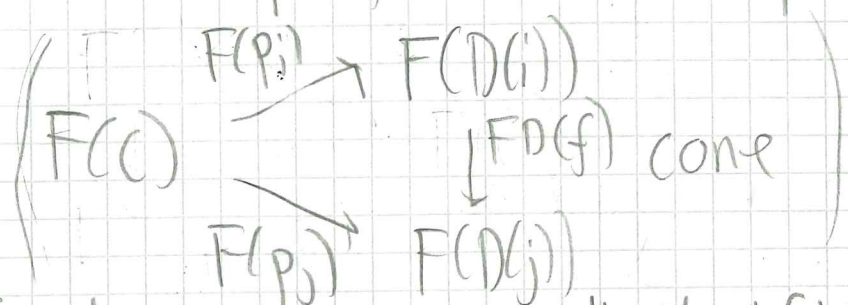
exists unique $\psi: Q \rightarrow A$ s.t. commutes



Def. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ functor and \mathcal{I} a small cat.

F preserves limits of shape \mathcal{I} if \forall functors $D: \mathcal{I} \rightarrow \mathcal{C}$

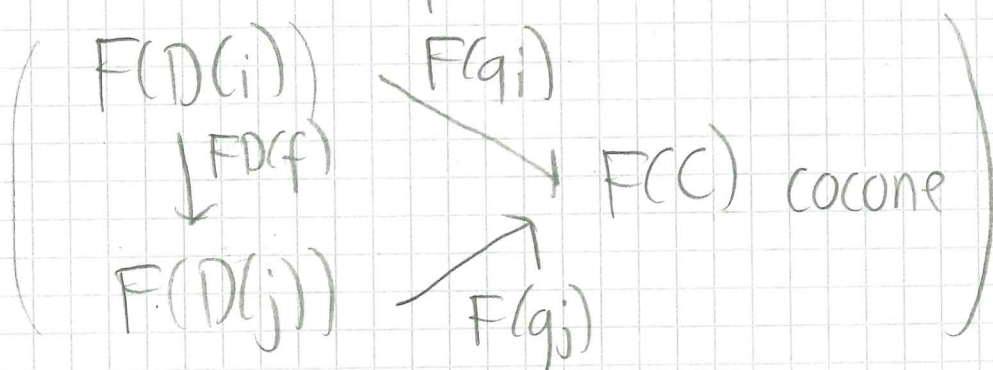
if (C, p_i) is a limit of D , then
 $(F(C), F(p_i))$ is a limit of $F \circ D$



Simply say F preserves limits if it preserves limits of all shapes

• F preserves colimits of shape \mathcal{I}

if \forall functors $D: \mathcal{I} \rightarrow \mathcal{C}$, if (C, q_i) is a colimit of D , then $(F(C), F(q_i))$ is a colimit of $F \circ D$



F preserves colimits if it preserves colimits of all shapes

Theorem $F: \mathcal{C} \rightarrow \mathcal{D}$, $G: \mathcal{D} \rightarrow \mathcal{C}$ functors,
 (F, G) adjoint pair. Then
 F preserves colimits & G preserves limits.