Example (Free modules) R ring

Forgetful functor $F: \mathrm{Mod}\,R \to \mathrm{Set}$ has a left adjoint $R^{(-)}: \mathrm{Set} \to \mathrm{Mod}\,R$ given as follows

$$\underset{\text{set}}{X} \longmapsto R^{(X)} = \left\{ \text{maps } f: X \to R \;\middle|\; \begin{array}{l} f(x) \neq 0 \text{ for only} \\ \text{finitely many } x \in X \end{array} \right\}$$

$\varphi: X \to Y$ maps in Set. Then

$R^{(\varphi)}: R^{(X)} \to R^{(Y)}$ given by

$$R^{(\varphi)}(f): Y \to R$$

$$y \longmapsto \sum_{x \in \varphi^{-1}(y)} f(x)$$

The adjunction iso is

$$\mathrm{Hom}_R(R^{(X)}, M) \longrightarrow \mathrm{Hom}_{\mathrm{Set}}(X, F(M))$$

$$\psi \longmapsto [x \mapsto \psi(\chi_x)]$$

$$\Big[f \mapsto \sum_{x \in X} \overset{\in R}{f(x)} \overset{\in M}{\varphi(x)}\Big] \longleftarrow \varphi$$

where $\chi_x: X \to R$, $\chi_x(y) = \begin{cases} 0 & x \neq y \\ 1_R & x = y \end{cases}$

Proposition (Unit-counit adjunction)

$F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ functors.

The following are equivalent:

1. (F, G) is an adjoint pair.
2. $\exists$ natural transformations

$\eta: Id_{\mathcal{C}} \to G \circ F$ & $\varepsilon: F \circ G \to Id_{\mathcal{D}}$

s.t. $\forall X \in \mathcal{C}$, $Y \in \mathcal{D}$ the diagrams

$$\begin{tikzcd} F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "id_{F(X)}"'] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \\ & F(X) \end{tikzcd} \qquad \begin{tikzcd} G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "id_{G(Y)}"'] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] \\ & G(Y) \end{tikzcd}$$

commutes.

Sketch of proof:

Let $\phi_{X,Y}: Hom_{\mathcal{D}}(F(X), Y) \to Hom_{\mathcal{C}}(X, G(Y))$ adjunction iso. Define

$\eta_X = \phi_{X,F(X)}(id_{F(X)}): X \to GF(X)$

$\varepsilon_Y = \phi^{-1}_{G(Y),Y}(id_{G(Y)}): FG(Y) \to Y$.

Check:

- $\eta = \{\eta_X \mid X \in \mathcal{C}\}$ & $\varepsilon = \{\varepsilon_Y \mid Y \in \mathcal{D}\}$ are natural transformations

- $\varepsilon_{F(X)} \circ F(\eta_X) = id_{F(X)}$ & $G(\varepsilon_Y) \circ \eta_{G(Y)} = id_{G(Y)}$.

Conversely, let $\eta : id_{\mathcal{C}} \to G \circ F$ and $\varepsilon : F \circ G \to id_{\mathcal{D}}$ be as in (2).

Define

$$\phi_{XY} : Hom_{\mathcal{D}}(F(X), Y) \xrightarrow{G(-)} Hom_{\mathcal{C}}(GF(X), G(Y)) \xrightarrow{Hom_{\mathcal{C}}(\eta_X, G(Y))} Hom_{\mathcal{C}}(X, G(Y))$$

Check: • $\phi_{X,Y}$ is an iso $\forall X \in \mathcal{C}, Y \in \mathcal{D}$

• $\phi = \{\phi_{X,Y}\}$ is natural.

Limits

__Def:__ $\mathcal{I}$ small cat (Ob $\mathcal{I}$ is a set), $\mathcal{C}$ cat, $D : \mathcal{I} \to \mathcal{C}$ functor.

- A <u>cone</u> (C, q_i) of D is an object $C \in \mathcal{C}$ together with morphisms $\{q_i : C \to D(i) \mid i \in \mathcal{I}\}$ s.t.

$$\begin{tikzcd} & D(i) \arrow[d, "D(f)"] \\ C \arrow[ur, "q_i"] \arrow[r, "q_j"'] & D(j) \end{tikzcd}$$

commutes $\forall$ morphisms f in $\mathcal{I}$

- A <u>limit</u> (C, p_i) of D is a cone such that for any other cone (C', p_i') of D there exists a <u>unique</u> morphism $\varphi : C' \to C$ s.t.

$$\begin{tikzcd} C' \arrow[rr, "\varphi"] \arrow[dr, "p_i'"'] & & C \arrow[dl, "p_i"] \\ & D(i) & \end{tikzcd}$$

commutes $\forall i \in \mathrm{Ob}\,\mathcal{I}$.

write $C = \varprojlim D$

Dual definitions using $\mathcal{C}^{op}$:

- A <u>cocone</u> (C, q_i) of D is
  - $C \in \mathcal{C}$
  - $q_i : D(i) \to D$ s.t.

$$\begin{tikzcd} D(i) \arrow[dr, "q_i"] \arrow[d, "D(f)"'] & \\ D(j) \arrow[r, "q_j"'] & D \end{tikzcd}$$

commutes $\forall f$.

- A <u>colimit</u> of D is a cocone (C, q_i) s.t. $\forall$ cocones (C', q_i') $\exists$ unique morphism $\psi : C \to C'$ s.t.

$$\begin{tikzcd} C \arrow[rr, "\psi"] & & C' \\ & D(i) \arrow[ul, "q_i"] \arrow[ur, "q_i'"'] & \end{tikzcd}$$

commutes $\forall i \in \mathcal{I}$.

$C = \varinjlim D$

<u>Remark:</u>

- In the notes limits/colimits are defined for contravariant functors $\mathcal{I}^{\circ} \to \mathcal{C}$.
- limits & colimits are unique if they exist (check!)

Ex (product): X set, $\{C_x \mid x \in X\}$ set of objects in $\mathcal{C}$.

Define cat $\mathcal{C}_X$:

$\mathrm{Obj}\, \mathcal{C}_X = X$

$$\mathrm{Hom}_{\mathcal{C}_X}(x,y) = \begin{cases} \emptyset & x \neq y \\ \{id_x\} & x = y \end{cases}$$

$x \longmapsto C_x$ defines a functor

$D : \mathcal{C}_X \longrightarrow \mathcal{C}$.

- If $\varprojlim D$ exists, it is called the <u>product</u> of $\{C_x\}$, denoted $\prod_{x \in X} C_x$
- If $\varinjlim D$ exists, it is called the <u>coproduct</u> of $\{C_x\}$, denoted $\coprod_{x \in X} C_x$

For example, for $C, C' \in \mathcal{C}$:

product

$$\begin{tikzcd}
 & \forall A \arrow[dl, "\forall p_2'"'] \arrow[dr, "\forall p_1'"] \arrow[d, dashed, "\exists! \varphi"] & \\
C_1 & C_1 \prod C_2 \arrow[l, "p_1"] \arrow[r, "p_2"'] & C_2
\end{tikzcd}$$

coproduct

$$\begin{tikzcd}
 & C_1 \amalg C_2 \arrow[d, dashed, "\exists! \psi"] & \\
C_1 \arrow[ur, "q_1"] \arrow[r, "\forall q_1'"'] & \forall A & C_2 \arrow[ul, "q_2"'] \arrow[l, "\forall q_2'"]
\end{tikzcd}$$

For Set

products = cartesian products

coproducts = disjoint unions. (check!)

<u>Example</u> (pullbacks & pushouts)

(1) Let $\mathcal{I}$ be the cat

i.e. Ob $\mathcal{I}$ = $\{1, 2, 3\}$

$$\begin{tikzcd}
 & 2 \arrow[d, "v"] \\
1 \arrow[r, "u"] & 3
\end{tikzcd}$$

$\mathrm{Hom}_{\mathcal{I}}(1,2) = \{u\}$ $\quad \mathrm{Hom}_{\mathcal{I}}(1,3) = \emptyset$

$\mathrm{Hom}_{\mathcal{I}}(1,1) = \{id_1\}$ ...

A functor $D: \mathcal{I} \to \mathcal{C}$ is given by a diagram

$$\begin{tikzcd} & C_2 \arrow[d, "\beta"] \\ C_1 \arrow[r, "\alpha"] & C_3 \end{tikzcd}$$

$D(i) = C_i$
$D(u) = \alpha$
$D(v) = \beta$

The limit of D is called a <u>pullback</u> of the diagram. Given by

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & C_2 \arrow[d, "\beta"] \\ C_1 \arrow[r, "\alpha"] & C_3 \end{tikzcd}$$

commutative diagram (sometimes called <u>cartesian</u>)

s.t. $\forall A \in \mathcal{C}$ & commutative diagrams

$$\begin{tikzcd} A \arrow[r, "p_2'"] \arrow[d, "p_1'"'] & C_2 \arrow[d, "\beta"] \\ C_1 \arrow[r, "\alpha"'] & C_3 \end{tikzcd}$$

$\exists$ unique map $\varphi: A \to P$ s.t commutes.

$$\begin{tikzcd}
A \arrow[dr, "\varphi"] \arrow[drr, bend left, "p_2'"] \arrow[ddr, bend right, "p_1'"'] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\circ"] & C_2 \arrow[d] \\
& C_1 \arrow[r] & C_3
\end{tikzcd}$$

(what happens to p_3 & q_3?)

(2)

$$I = \begin{tikzcd}
1 \arrow[r] \arrow[d] & 2 \\
3 &
\end{tikzcd}$$

$$D: I \to \mathcal{C} \iff \text{diagram} \quad \begin{tikzcd}
C_3 \arrow[r, "\alpha"] \arrow[d, "\beta"'] & C_2 \\
C_1 &
\end{tikzcd}$$

A colimit of F is called a <u>pushout</u>

Given by comm diagram (sometimes called <u>cocartesian</u>)

$$\begin{tikzcd}
C_3 \arrow[r, "\alpha"] \arrow[d, "\beta"'] & C_2 \arrow[d, "q_2"] \\
C_1 \arrow[r, "q_1"'] & Q
\end{tikzcd}$$

s.t. for all comm diagram

$$\begin{tikzcd}
C_3 \arrow[r, "\alpha"] \arrow[d, "\beta"'] & C_2 \arrow[d, "q_2'"] \\
C_1 \arrow[r, "q_1'"'] & A
\end{tikzcd}$$

exists unique $\psi: Q \to A$ s.t. commutes

$$\begin{tikzcd}
C_3 \arrow[r, "\alpha"] \arrow[d, "\beta"'] & C_2 \arrow[d, "q_{2_0}"] \arrow[ddr, bend left, "q_2'"] & \\
C_1 \arrow[r, "q_{1_0}"'] \arrow[drr, bend right, "q_1'"'] & Q \arrow[dr, "\psi"] & \\
& & A
\end{tikzcd}$$

<u>Def</u>: Let $F: \mathcal{C} \to \mathcal{D}$ functor and $\mathcal{I}$ a small cat.

- F <u>preserves limits of shape $\mathcal{I}$</u> if $\forall$ functors $D: \mathcal{I} \to \mathcal{C}$

if (C, p_i) is a limit of D, then
$(F(C), F(p_i))$ is a limit of $F \circ D$

$$\left(\begin{tikzcd} & F(D(i)) \arrow[d, "FD(f)"] \\ F(C) \arrow[ur, "F(p_i)"] \arrow[r, "F(p_j)"'] & F(D(j)) \end{tikzcd} \text{ cone}\right)$$

Simply say F <u>preserves limits</u> if it preserves limits of all shapes

• F <u>preserves colimits of shape $\mathcal{I}$</u>
if $\forall$ functors $D: \mathcal{I} \to \mathcal{C}$, if (C, q_i) is a colimit of D, then $(F(C), F(q_i))$ is a colimit of $F \circ D$

$$\left(\begin{tikzcd} F(D(i)) \arrow[d, "FD(f)"] \arrow[r, "F(q_i)"] & F(C) \\ F(D(j)) \arrow[ur, "F(q_j)"'] & \end{tikzcd} \text{ cocone}\right)$$

F <u>preserves colimits</u> if it preserves colimits of all shapes

<u>Theorem</u> $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ functors, (F, G) adjoint pair. Then F preserves colimits & G preserves limits.