- $f^\bullet$ is a homotopy equivalence in $Ch(\mathcal{A})$ iff $f^\bullet$ is an isomorphism in $K(\mathcal{A})$

Lecture 14

Lemma: $\mathcal{A}$ abelian category. The following hold:

(1) If $f^\bullet : A^\bullet \to B^\bullet$ is nullhomotopic, then $H^n(f^\bullet) = 0 \ \forall n \in \mathbb{Z}$.

(2) H^n descends to a functor $H^n(-) : K(\mathcal{A}) \to \mathcal{A}$

$$\left(\begin{tikzcd} Ch(\mathcal{A}) \arrow[d] \arrow[dr, "H^n(-)"] & \\ K(\mathcal{A}) \arrow[r, "H^n(-)"'] & Ab \end{tikzcd}\right)$$

(3) If $f^\bullet$ is a homotopy equivalence, then $f^\bullet$ is a quasi-isomorphism.

Proof:

(1) (For Mod R): Have $f^n = d_B^{n-1} h^n + h^{n+1} d_A^n$ for morphisms $h^n : A^n \to B^{n-1}$. Hence for $x \in Z^n(A)$, have

$f^n(x) = d_B^{n-1} h^n(x) + h^{n+1} \underbrace{d_A^n(x)}_{=0} = d_B^{n-1} h^n(x)$, so

$H^n(f)(x + \operatorname{im} d_A^{n-1}) = f(x) + \operatorname{im} d_B^{n-1} = d_B^{n-1}(h^n(x)) + \operatorname{im} d_B^{n-1} = 0$

so $H^n(f)=0$

(2) If $f^\bullet \sim g^\bullet$, then $f^\bullet - g^\bullet$ is null-homotopic, so $0 = H^n(f^\bullet - g^\bullet) = H^n(f^\bullet) - H^n(g^\bullet)$. Hence, $H^n(f^\bullet) = H^n(g^\bullet)$. This implies the result.

(3) If $f^\bullet$ is a homotopy equivalence, then it is an isomorphism in $K(\mathcal{A})$. Since $H^n(-)$ is a functor: $K(\mathcal{A}) \to \mathcal{A}$, it must preserve isomorphisms in $K(\mathcal{A})$. Hence, if $f^\bullet$ is a homotopy equivalence, then $H^n(f)$ is an isomorphism $\forall n \in \mathbb{Z}$.

$\Rightarrow f^\bullet$ is a quasi-isomorphism ∎

<u>Projective and injective resolutions</u>

<u>Def</u>: $\mathcal{A}$ abelian cat.

(1) $\mathcal{A}$ has <u>enough projectives</u> if $\forall A \in \mathcal{A}$ there exists $P \in \mathcal{A}$ projective and an epimorphism $P \twoheadrightarrow \mathcal{A}$

(2) $\mathcal{A}$ has <u>enough injectives</u> if $\forall A \in \mathcal{A}$ there exists $I \in \mathcal{A}$ injective and a monomorphism $A \to I$

Ex: (1) Mod R has enough projectives and injectives

(2) mod $\mathbb{Z}$ - category of fin. gen abelian groups. Then mod $\mathbb{Z}$ has enough projectives, but not enough injectives

(3) There exist abelian categories with no injective or projective object, e.g. the category of coherent sheaves on the projective line $\mathbb{P}^1_{\mathbb{C}}$

Def: $\mathcal{A}$ abelian cat, and $A \in \mathcal{A}$.

(1) A projective resolution of A is a complex

$$P^\cdot : \cdots \to P^{-2} \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^0 \to 0 \to 0 \to \cdots$$

with projective terms, which is exact everywhere except in position 0, where $H^0(P^\cdot) = \operatorname{Coker} d^{-1} = A$

(2) An injective resolution of A is a complex

$$\cdots \to 0 \to 0 \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \to \cdots$$

with injective terms, which is exact everywhere except in position 0, where $H^0(I^\cdot) = \operatorname{Ker} d^0 = A$

<u>Construction</u>: Let $\mathcal{A}$ be an abelian cat with enough projectives. Then any $A \in \mathcal{A}$ has a projective resolution, constructed in the following way:

Let $0 \longrightarrow K^{-1} \overset{i_{-1}}{\rightarrowtail} P^0 \overset{p_0}{\longrightarrow} A \to 0$ exact with P_0 projective

$$
\begin{tikzcd}
 & & & & K^{-2} \arrow[dr, tail, "i_{-2}"] & & & \\
P^{\bullet} = \cdots & P^{-3} \arrow[rr, "d^{-3}"] \arrow[dr, two heads, "p_{-3}"'] & & P^{-2} \arrow[rr, "d^{-2}"] \arrow[ur, two heads, "p_{-2}"] & & P^{-1} \arrow[rr, "d^{-1}"] \arrow[dr, two heads, "p_{-1}"'] & & P^0 \\
 & & K^{-3} \arrow[ur, tail, "i_{-3}"'] & & & & K^{-1} \arrow[ur, tail, "i_{-1}"'] &
\end{tikzcd}
$$

Iterative choose exact sequences
$0 \to K^{-n-1} \xrightarrow{i_{-n-1}} P^{-n} \xrightarrow{p_{-n}} K^{-n} \to 0$ with P^{-n} projective, and set $d^{-n} = i_{-n} \circ p_{-n}$. Then $P^{\bullet}$ is a proj resolution of A.

If $\mathcal{A}$ has enough injectives, the dual construction works to get an injective coresolution of A.

Lemma: $\mathcal{A}$ abelian cat with enough projectives.

(1) Any object $A \in \mathcal{A}$ has a projective resolution

(2) Let $f: A \to B$ morphism in $\mathcal{A}$, and let $P^{\cdot}$ and $Q^{\cdot}$ be projective resolutions of A and B, respectively. Then f can be lifted to a morphism $f^{\cdot}: P^{\cdot} \to Q^{\cdot}$ of chain complexes, i.e. there exists morphisms $f^{-n}: P^{-n} \to Q^{-n}$ s.t.

$$\begin{tikzcd}
\cdots \arrow[r] & P^{-2} \arrow[r, "d_P^{-2}"] \arrow[d, "f^{-2}"] & P^{-1} \arrow[r, "d_P^{-1}"] \arrow[d, "f^{-1}"] & P^{0} \arrow[r] \arrow[d, "f^{0}"] & A \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & Q^{-2} \arrow[r, "d_Q^{-2}"] & Q^{-1} \arrow[r, "d_Q^{-1}"] & Q^{0} \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

commutes.

(3) The lifting in (2) is unique up to homotopy, i.e. if $\tilde{f}^{\cdot}: P^{\cdot} \to Q^{\cdot}$ is another lift of f, then $\tilde{f}^{\cdot} \sim f^{\cdot}$

Proof

(1) Follows from the construction above.

(2) Note that any exact sequence $X^\bullet$ can be constructed via combining short exact sequences, since $\operatorname{im} d^{n+1} = \operatorname{Ker} d^n$.

$$
\begin{tikzcd}
\operatorname{Im} d^{n-2} \arrow[dr, tail] & & & \operatorname{Im} d^{n} \arrow[dr, tail] & & & \\
\cdots \arrow[r] & X^{n-1} \arrow[rr, "d^{n-1}"] \arrow[dr, two heads] & & X^{n} \arrow[rr, "d^{n}"] \arrow[u, two heads] & & X^{n+1} \arrow[r, "d^{n+1}"] \arrow[dr, two heads] & \cdots \\
& & \operatorname{Im} d^{n-1} \arrow[ur, tail] & & & & \operatorname{Im} d^{n+1}
\end{tikzcd}
$$

Here $0 \to \operatorname{Im} d^{i-1} \rightarrowtail X^{i} \twoheadrightarrow \operatorname{Im} d^{i} \to 0$

exact $\forall i \in \mathbb{Z}$. Using this, consider

$$
\begin{tikzcd}
& & \operatorname{im} d_P^{-2} \arrow[dr, tail] \arrow[ddd, dashed] & & & & & \\
\cdots \arrow[r] & P^{-2} \arrow[rr, "d_P^{-2}"] \arrow[ur, two heads, "p_{-2}"] \arrow[ddd, dashed] & & P^{-1} \arrow[rr, "d_P^{-1}"] \arrow[dr, two heads, "p_{-1}"] \arrow[ddd, dashed, "f^{-1}"] & & P^{0} \arrow[r, "p_0"] \arrow[ddd, dashed, "f^{0}"] & A \arrow[r] \arrow[ddd, "f"] & 0 \\
& & & & \operatorname{im} d_P^{-1} \arrow[ur, tail] \arrow[ddd, dashed, "g^{0}"] & & & \\
& & \operatorname{im} d_Q^{-2} \arrow[dr, tail] & & & & & \\
\cdots \arrow[r] & Q^{-2} \arrow[rr] \arrow[ur, two heads, "q_{-2}"] & & Q^{-1} \arrow[rr] \arrow[dr, two heads, "q_{-1}"'] & & Q^{0} \arrow[r, "q_0"] & B \arrow[r] & 0 \\
& & & & \operatorname{im} d_Q^{-1} \arrow[ur, tail] & & &
\end{tikzcd}
$$

f^0 exists, since P^0 projective & q_0 epi.

g^0 exists by commutativity of the right hand square.

f^{-1} is a lift of the map $g^0 \circ p_{-1}$, which exists since P^{-1} projective & q_{-1} epi

g^{-1} exists by commutativity of square with maps $p_{-1}, g^0, q_{-1}, f^{-1}$.

repeating this procedure, we get maps $f^{-n} : P^{-n} \to Q^{-n}$ which forms a morphism $f^\bullet : P^\bullet \to Q^\bullet$ in $Ch(\mathcal{A})$ lifting f

To prove (3) it suffices to show that $g^\bullet := f^\bullet - \tilde{f}^\bullet$ is null-homotopic, i.e. that any lift of the zero map is nullhomotopic. So consider

$$\begin{tikzcd}
\cdots \arrow[r] & P^{-2} \arrow[rr, "d_P^{-2}"] \arrow[dd, "g^{-2}"] & & P^{-1} \arrow[rr, "d_P^{-1}"] \arrow[dd, "g^{-1}"] \arrow[ddll, dashed, "h^{-1}"'] \arrow[dddl, dashed, "k^{-1}"] & & P^0 \arrow[r] \arrow[dd, "g^0"] \arrow[ddll, dashed, "h^0"'] \arrow[dddl, dashed, "k^0"] & A \arrow[dd, "0"] \\
& & & & & & \\
\cdots \arrow[r] & Q^{-2} \arrow[rr] \arrow[dr, "q_{-2}"'] & & Q^{-1} \arrow[rr] \arrow[dr, "q_{-1}"'] & & Q^0 \arrow[r, "q_0"] & B \\
& & \mathrm{im}\, d_Q^{-2} \arrow[ur, "j_{-2}"'] & & \mathrm{im}\, d_Q^{-1} \arrow[ur, "j_{-1}"'] & &
\end{tikzcd}$$

Since $q_0 \circ g^0 = 0$, the map g^0 factors through the kernel $\operatorname{imd}_Q^{-1}$ of q_0 via a map $k^0 : P^0 \longrightarrow \operatorname{imd}_Q^{-1}$.
Since $q_{-1} : Q^{-1} \longrightarrow \operatorname{imd}_Q^{-1}$ is an epimorphism & P^0 is projective, we can find lift $h^0 : P^0 \longrightarrow Q^{-1}$ satisfying $q_{-1} \circ h^0 = k^0$
Next consider the difference

$$g^{-1} - h^0 \circ d_P^{-1} : P^{-1} \to Q^{-1}.$$

Since $q_{-1} \circ (g^{-1} - h^0 \circ d_P^{-1}) = q_{-1} \circ g^{-1} - k^0 \circ d_P^{-1}$

and $j_{-1} \circ (q_{-1} \circ g^{-1} - k^0 \circ d_P^{-1})$

$$= (j_{-1} \circ q_{-1}) \circ g^{-1} - (j_{-1} \circ k^0) \circ d_P^{-1} = d_Q^{-1} \circ g^{-1} - g^0 \circ d_P^{-1} = 0$$

and j_{-1} is a monomorphism, it follows that $q_{-1} \circ (g^{-1} - h^0 \circ d_P^{-1}) = 0$, so $g^{-1} - h^0 \circ d_P^{-1}$ factors through $\operatorname{imd}_Q^{-2}$ via a morphism $k^{-1} : P^{-1} \longrightarrow \operatorname{imd}_Q^{-2}$.
Again, since P^{-1} is projective & $q_{-2} : Q^{-2} \longrightarrow \operatorname{imd}_Q^{-2}$ is an epimorphism, we can find a morphism $h^{-1} : P^{-1} \longrightarrow Q^{-2}$ s.t.

$$q_{-2} \circ h^{-1} = k^{-1}$$

It follows that $d_Q^{-2} \circ h^{-1} = g^{-1} - h^0 \circ d_P^{-1}$

$$\Leftrightarrow g^{-1} = d_Q^{-2} \circ h^{-1} + h^0 \circ d_P^{-1}$$

repeating this procedure, we can construct maps $h^{-2}: P^{-2} \to Q^{-3}$, $h^{-3}: P^{-3} \to Q^{-4}$, ...

giving a null-homotopy of $g^{\cdot}$ ▪

<u>Corollary</u>: Let $\mathcal{A}$ abelian cat with enough projectives. Then taking projective resolutions defines a functor

$$p: \mathcal{A} \to K(\mathcal{A})$$

such that $H^0 \circ p = id_{\mathcal{A}}$ and $H^n \circ p = 0$ $n \neq 0$.

<u>Proof</u>: This follows from the previous result ▪

The dual result gives a functor $i: \mathcal{A} \to K(\mathcal{A})$ where iA - injective resolution of A, satisfying $H^0 \circ i = id_{\mathcal{A}}$ & $H^n \circ i = 0$ $\forall n \neq 0$.

<u>Lemma</u> (Horseshoe lemma) $\mathcal{A}$ abelian cat, and let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{A}$. Assume $P^{\cdot}$ and $Q^{\cdot}$ are projective resolutions of A and C respectively. Then there exists a proj resolution $R^{\cdot}$ of B with $R^n = P^n \oplus Q^n$ $\forall n$, s.t. the following diagram commutes