

- f is a homotopy equivalence in $\text{Ch}(\mathcal{A})$
 iff f is an isomorphism in $K(\mathcal{A})$

Lecture 14

Lemma: \mathcal{A} abelian category. The following hold:

(1) If $f: A \rightarrow B$ is nullhomotopic, then
 $H^n(f) = 0 \forall n \in \mathbb{Z}$.

(2) H^n descends to a functor $H^n(-): K(\mathcal{A}) \rightarrow \mathcal{A}$

$$\begin{array}{ccc} \text{Ch}(\mathcal{A}) & \xrightarrow{H^n(-)} & \mathcal{A} \\ \downarrow \circ & \searrow & \\ K(\mathcal{A}) & \xrightarrow{H^n(-)} & \mathcal{A} \end{array}$$

(3) If f is a homotopy equivalence, then
 f is a quasi-isomorphism.

Proof:

(1) (For $\text{Mod } R$): Have $f = d_B^n h^{n-1} + h^n d_A^n$
 for morphisms $h: A \rightarrow B$. Hence for
 $x \in Z^n(A)$, have

$$\begin{aligned} f^n(x) &= d_B^{n-1} h^n(x) + h^n d_A^n(x) = d_B^{n-1} h^n(x) \\ H^n(f)(x + \text{im } d_A^{n-1}) &= f^n(x) + \text{im } d_B^{n-1} = d_B^{n-1} h^n(x) + \text{im } d_B^{n-1} = 0 \end{aligned}$$

so

$$\text{so } H^n(f) = 0$$

(2) If $f \sim g$, then $f - g$ is null-homotopic, so $0 = H^n(f - g) = H^n(f) - H^n(g)$. Hence, $H^n(f) = H^n(g)$. This implies the result.

(3) If f is a homotopy equivalence, then it is an isomorphism in $K(\mathcal{A})$.

Since $H^n(-)$ is a functor $: K(\mathcal{A}) \rightarrow \mathcal{A}$,

it must preserve isomorphisms in $K(\mathcal{A})$.

Hence, if f is a homotopy equivalence, then $H^n(f)$ is an isomorphism $\forall n \in \mathbb{Z}$.

$\Rightarrow f$ is a quasi-isomorphism.

Projective and injective resolutions

Def. \mathcal{A} abelian cat.

(1) \mathcal{A} has enough projectives if $\forall A \in \mathcal{A}$ there exists $P \in \mathcal{A}$ projective and an epimorphism $P \twoheadrightarrow A$.

(2) \mathcal{A} has enough injectives if $\forall A \in \mathcal{A}$ there exists $I \in \mathcal{A}$ injective and a monomorphism $A \rightarrow I$.

Ex: (1) $\text{Mod } R$ has enough projectives and injectives

(2) $\text{mod } \mathbb{Z}$ -category of fin. gen abelian groups, $\text{Ther mod } \mathbb{Z}$ has enough projectives, but not enough injectives

(3) There exist abelian categories with no injective or projective object, e.g. the category of coherent sheaves on the projective line $\mathbb{P}_{\mathbb{C}}^1$

Def: \mathcal{A} abelian cat, and $A \in \mathcal{A}$.

(1) A projective resolution of A is a complex

$$P: \dots \rightarrow P^{-2} \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^0 \rightarrow 0 \rightarrow 0 \rightarrow \dots$$

with projective terms, which is exact everywhere except in position 0, where $H^0(P^\bullet) = \text{Coker } d^{-1} = A$

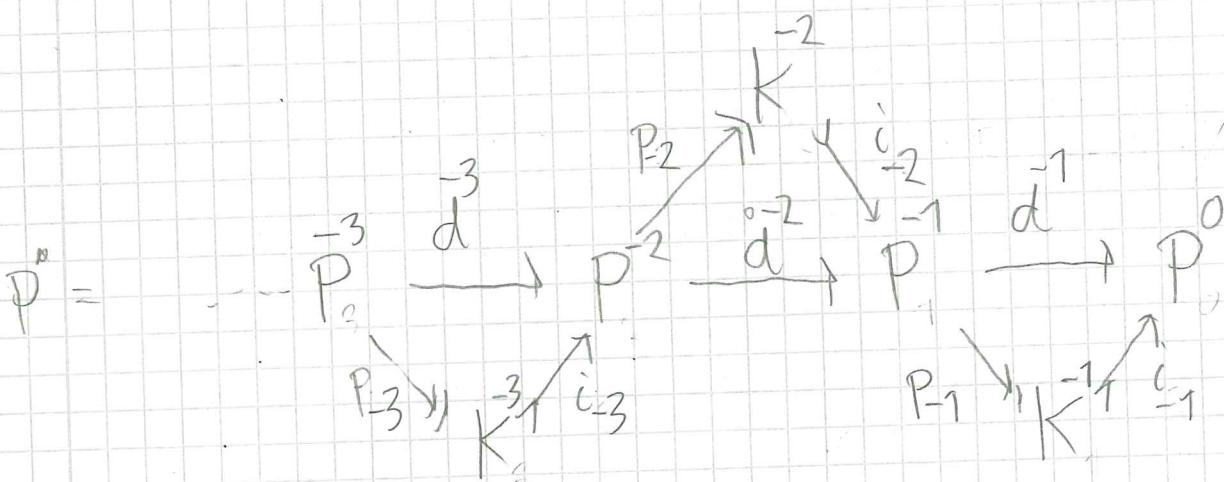
(2) An injective resolution of A is a complex

$$\dots \rightarrow 0 \rightarrow 0 \rightarrow I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \rightarrow \dots$$

with injective terms, which is exact everywhere except in position 0, where $H^0(I^\bullet) = \text{Ker } d^0 = A$

Construction: Let \mathcal{A} be an abelian cat with enough projectives. Then any $A \in \mathcal{A}$ has a projective resolution, constructed in the following way:

Let $0 \rightarrow K \xrightarrow{i_1} P_0 \xrightarrow{p_0} A \rightarrow 0$ exact with P_0 projective



Iterative choose exact sequences

$$0 \rightarrow K^{n-1} \xrightarrow{i_{n-1}} \tilde{P}^n \xrightarrow{p_n} K^n \rightarrow 0 \text{ with } \tilde{P}^n \text{ projective,}$$

and set $d^n = i_{n-1} \circ p_n$. Then P^\bullet is a proj resolution of A .

If \mathcal{A} has enough injectives, the dual construction works to get an injective coresolution of A .

Lemma: \mathcal{A} abelian cat with enough projectives.

(1) Any object $A \in \mathcal{A}$ has a projective resolution

(2) Let $f: A \rightarrow B$ morphism in \mathcal{A} , and let P^\bullet and Q^\bullet be projective resolutions of A and B , respectively. Then f can be lifted to a morphism

$f^\bullet: P^\bullet \rightarrow Q^\bullet$ of chain complexes, i.e.

there exists morphisms $f^{-n}: P^{-n} \rightarrow Q^{-n}$ s.t.

$$\begin{array}{ccccccc}
 \rightarrow & P^{-2} & \xrightarrow{d_P^{-2}} & P^{-1} & \xrightarrow{d_P^{-1}} & P^0 & \rightarrow A \rightarrow 0 \\
 & \downarrow f^{-2} & & \downarrow f^{-1} & & \downarrow f^0 & \downarrow f \\
 \rightarrow & Q^{-2} & \xrightarrow{d_Q^{-2}} & Q^{-1} & \xrightarrow{d_Q^{-1}} & Q^0 & \rightarrow B \rightarrow 0
 \end{array}$$

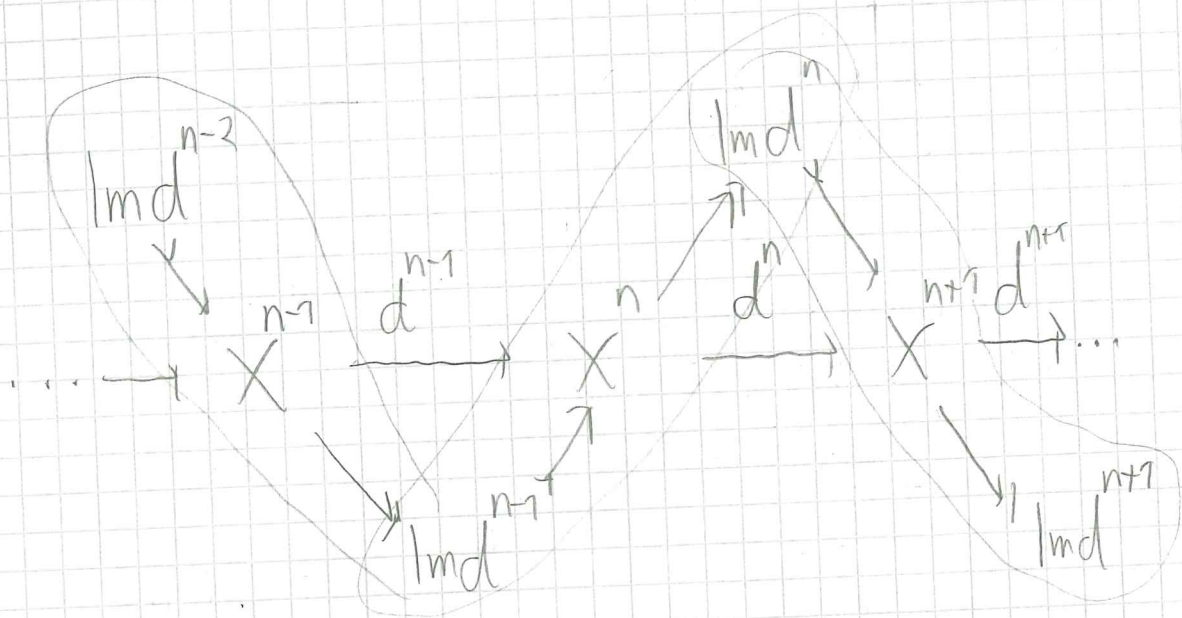
commutes.

(3) The lifting in (2) is unique up to homotopy, i.e. if $\tilde{f}^\bullet: P^\bullet \rightarrow Q^\bullet$ is another lift of f , then $\tilde{f}^\bullet \sim f^\bullet$.

Proof

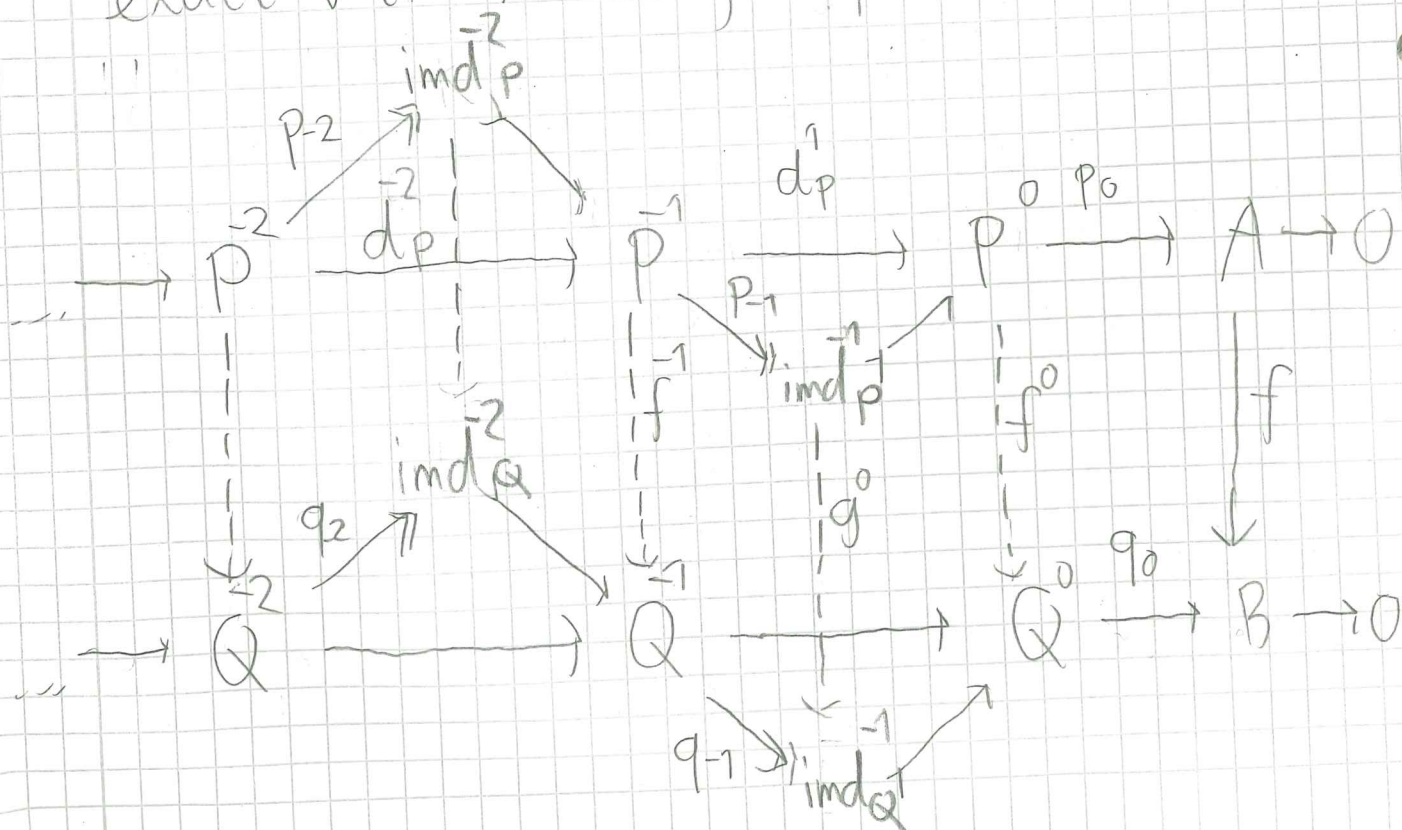
(1) Follows from the construction above.

(2) Note that any exact sequence X^\bullet can be constructed via combining short exact sequences, since $\text{im } d^{n+1} = \text{ker } d^n$.



Here $0 \rightarrow \text{im } d^{i-1} \rightarrow X^i \xrightarrow{d^i} \text{im } d^i \rightarrow 0$

exact $\forall i \in \mathbb{Z}$. Using this, consider



f^0 exists, since P^0 projective & q_0 epi.

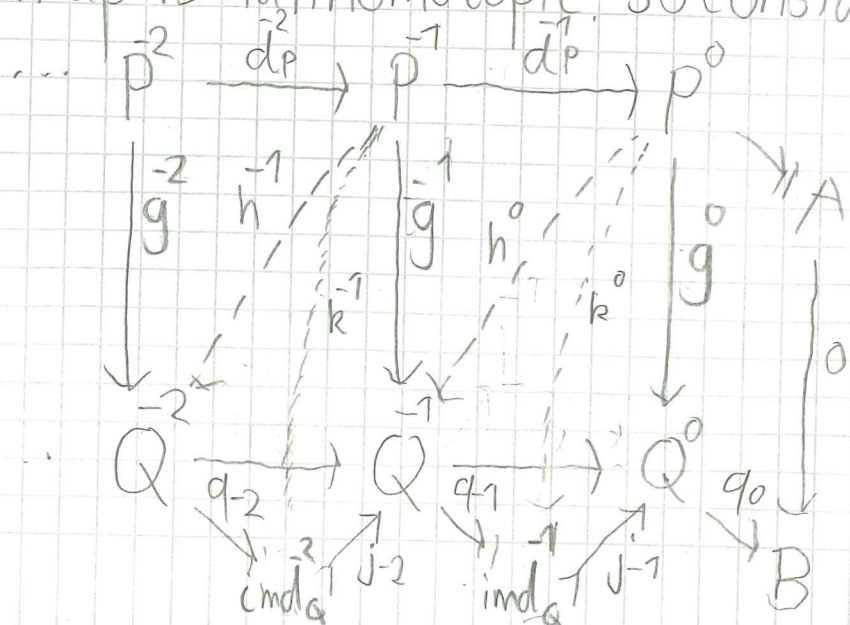
g^0 exists by commutativity of the right hand square.

$\xrightarrow{-1}$
 f is a lift of the map $g^0 \circ p_{-1}$, which exists since P^{-1} projective & q_{-1} epi.

$\xrightarrow{-1}$
 g exists by commutativity of square with maps $p_{-1}, g^0, q_{-1}, f^{-1}$.

repeating this procedure, we get maps $f^{-n}: P^{-n} \rightarrow Q^{-n}$ which forms a morphism $f: P^0 \rightarrow Q^0$ in $\text{Ch}(\mathcal{A})$ lifting f

To prove (3) it suffices to show that $g \circ f - \tilde{f}$ is null-homotopic, i.e. that any lift of the zero map is null-homotopic. So consider



Since $q_0 \circ g^0 = 0$, the map g^0 factors through the kernel $\text{ker } q_0$ of q_0 via a map $k^0: P^0 \rightarrow \text{ker } q_0$. Since $q_{-1}: Q^0 \rightarrow \text{ker } q_0$ is an epimorphism & P_{-1}^0 is projective, we can find lift $h^0: P^0 \rightarrow Q^0$ satisfying $q_{-1}^0 \circ h^0 = k^0$.

Next consider the difference

$$\bar{g}^{-1} - h^0 \circ d_P^{-1}: P^{-1} \rightarrow Q^{-1}$$

$$\text{Since } q_{-1}^0 \circ (\bar{g}^{-1} - h^0 \circ d_P^{-1}) = q_{-1}^0 \bar{g}^{-1} - k^0 \circ d_P^{-1}$$

$$\text{and } j_{-1}^0 \circ (q_{-1}^0 \bar{g}^{-1} - k^0 \circ d_P^{-1})$$

$$= (j_{-1}^0 \circ q_{-1}^0) \circ \bar{g}^{-1} - (j_{-1}^0 \circ k^0) \circ d_P^{-1} = d_Q^0 \circ \bar{g}^{-1} - g^0 \circ d_P^{-1} = 0$$

and j_{-1}^0 is a monomorphism, it follows that

$$q_{-1}^0 \circ (\bar{g}^{-1} - h^0 \circ d_P^{-1}) = 0, \text{ so } \bar{g}^{-1} - h^0 \circ d_P^{-1} \text{ factors}$$

through $\text{ker } q_{-1}^0$ via a morphism $k^1: P^{-1} \rightarrow \text{ker } q_{-1}^0$.

Again, since P^{-1} is projective & $q_{-2}: Q^{-1} \rightarrow \text{ker } q_{-1}^0$ is an epimorphism, we can find a morphism

$$h^{-1}: P^{-1} \rightarrow Q^{-1} \text{ s.t. } q_{-2}^{-1} \circ h^{-1} = k^1$$

It follows that $d_Q^{-1} \circ h^{-1} = \bar{g}^{-1} - h^0 \circ d_P^{-1}$

$$\Leftrightarrow \bar{g}^{-1} = d_Q^{-1} \circ h^{-1} + h^0 \circ d_P^{-1}$$

repeating this procedure, we can construct maps $\overset{-2}{h}: \overset{-2}{P} \rightarrow \overset{-3}{Q}$, $\overset{-3}{h}: \overset{-3}{P} \rightarrow \overset{-4}{Q}$, ...

giving a null-homotopy of g .

Corollary: Let \mathcal{A} abelian cat with enough projectives. Then taking projective resolutions defines a functor

$$p: \mathcal{A} \rightarrow K(\mathcal{A})$$

such that $H^0 p = \text{id}_{\mathcal{A}}$ and $H^n p = 0 \quad n \neq 0$.

Proof: This follows from the previous result.

The dual result gives a functor $c: \mathcal{A} \rightarrow K(\mathcal{A})$ where cA -injective resolution of A , satisfying $H^0 c = \text{id}_{\mathcal{A}}$ & $H^n c = 0 \quad \forall n \neq 0$.

Lemma (Horseshoe lemma) \mathcal{A} abelian cat, and

let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a short exact sequence in \mathcal{A} . Assume P and Q are projective resolutions of A and C respectively. Then there exists a proj resolution R of B with

$R^n = P^n \oplus Q^n \quad \forall n$, s.t. the following diagram commutes