$\Leftrightarrow \exists$ morphism $0 \to Z$ s.t.

$$\begin{tikzcd} T \arrow[r] \arrow[d, "v"'] & 0 \arrow[d] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

commutes

$\Leftrightarrow g \circ v = 0$. Summarizing

$g \circ v = 0 \Leftrightarrow \exists u \in \mathrm{Hom}_{\mathcal{T}}(T, X)$ s.t. $v = f \circ u$.

This shows exactness. ■

<u>Lemma</u>: $f : X \to Y$ morphism in cat $\mathcal{C}$. The following are equivalent:

- f is an isomorphism
- $\mathrm{Hom}_{\mathcal{C}}(Z, X) \xrightarrow{f \circ -} \mathrm{Hom}_{\mathcal{C}}(Z, Y)$ bijective $\forall Z \in \mathcal{C}$,
- $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \xrightarrow{- \circ f} \mathrm{Hom}_{\mathcal{C}}(X, Z)$ bijective $\forall Z \in \mathcal{C}$.

<u>Pf</u>: Exercise ■

<u>Theorem</u>: (2 out of 3 property for isomorphisms)

$(\mathcal{T}, [1], \Delta)$ triang cat. Consider comm diagram

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d, "f"] & Y_1 \arrow[r] \arrow[d, "g"] & Z_1 \arrow[r] \arrow[d, "h"] & X_1[1] \arrow[d, "f[1]"] & \in \Delta \\ X_2 \arrow[r] & Y_2 \arrow[r] & Z_2 \arrow[r] & X_2[1] & \in \Delta \end{tikzcd}$$

If two out of f, g, and h are isomorphisms, then so is the third.

Pf: By rotation axiom (T2), may assume f & g isos. Applying $\mathrm{Hom}(-, W)$ with $W \in \mathcal{T}$ arbitrary, get commutative diagram

$$\begin{tikzcd}
(X_1, W) & (Y_1, W) \arrow[l] & (Z_1, W) \arrow[l] & (X_1[1], W) \arrow[l] & (Y_1[1], W) \arrow[l] & \cdots \arrow[l] \\
(X_2, W) \arrow[u, "-\circ f"] & (Y_2, W) \arrow[l] \arrow[u, "-\circ g"] & (Z_2, W) \arrow[l] \arrow[u, "-\circ h"] & (X_2[1], W) \arrow[l] \arrow[u, "-\circ f[1]"] & (Y_2[1], W) \arrow[l] \arrow[u, "-\circ g[1]"] & \cdots \arrow[l]
\end{tikzcd}$$

$(-, \cdot) := \mathrm{Hom}_{\mathcal{T}}(-, \cdot)$. Two leftmost and rightmost morphisms in diagram are isomorphisms since f, g isos

Five lemma $\Rightarrow -\circ h : (Z_2, W) \to (Z_1, W)$ iso $\forall W \in \mathcal{T}$

Previous lemma $\Rightarrow h : Z_1 \to Z_2$ isomorphism ∎

Homotopy categories are triangulated

$\mathcal{A}$ additive cat

$K(\mathcal{A})$ - homotopy cat

$[1] : Ch(\mathcal{A}) \to Ch(\mathcal{A})$ degree shift functor

$X[1]^n = X^{n+1}$ $\quad d^n_{X[1]} = -d^{n+1}_X$

$[1]$ preserves homotopy, so induces a functor

$[1] : K(\mathcal{A}) \to K(\mathcal{A})$.

<u>Recall</u>: For $f^\cdot : X^\cdot \to Y^\cdot$ in $Ch(\mathcal{A})$, have associate $Cone(f^\cdot) \in Ch(\mathcal{A})$

$$Cone(f^\cdot)^n = Y^n \oplus X^{n+1}, \quad d^n_{Cone(f^\cdot)} = \begin{pmatrix} d^n_Y & f^{n+1} \\ 0 & -d^{n+1}_X \end{pmatrix}$$

and a componentwise split exact seq

$$Y^\cdot \longrightarrow Cone(f^\cdot) \longrightarrow X^\cdot[1] \text{ in } Ch(\mathcal{A}).$$

Call sequence

$$X^\cdot \xrightarrow{f^\cdot} Y^\cdot \longrightarrow Cone(f^\cdot) \to X^\cdot[1]$$

<u>standard triangle</u>

Call a sequence $X^\cdot \to Y^\cdot \to Z^\cdot \to X^\cdot[1]$ in $K(\mathcal{A})$ a <u>triangle</u>.

Δ = all triangles isomorphic in $K(\mathcal{A})$ to a standard triangle

$$\begin{tikzcd}
X_1^\cdot \arrow[r] \arrow[d, "u"] & Y_1^\cdot \arrow[r] \arrow[d, "v"] & Z^\cdot \arrow[r] \arrow[d, "w"] & X_1^\cdot[1] \arrow[d, "u[1]"] \\
X_2^\cdot \arrow[r, "f^\cdot"] & Y_2^\cdot \arrow[r] & Cone(f^\cdot) \arrow[r] & X_2^\cdot[1]
\end{tikzcd}$$

comm in $K(\mathcal{A})$

u, v, w iso in $K(\mathcal{A})$

$\mathcal{A}$ - additive

<u>Goal</u>: Show $(K(\mathcal{A})[1], \Delta)$ is triangulated

<u>Lemma</u>: $(K(\mathcal{A}), [1], \Delta)$ satisfies (T2)

<u>Pf</u>: Only show left rotation

$(X^\bullet \xrightarrow{f} Y^\bullet \to Z^\bullet \to X^\bullet[1]) \in \Delta \Rightarrow (Y^\bullet \to Z^\bullet \to X^\bullet[1] \xrightarrow{-f[1]} Y^\bullet[1]) \in \Delta$

right rotation dual.

Suffices to show for standard triangle

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{i^\bullet} \mathrm{Cone}(f^\bullet) \xrightarrow{\pi^\bullet} X^\bullet[1], \text{ i.e.}$$

$$Y^\bullet \xrightarrow{i^\bullet} \mathrm{Cone}(f^\bullet) \xrightarrow{\pi^\bullet} X^\bullet[1] \xrightarrow{-f[1]} Y^\bullet[1] \in \Delta$$

suffices to show iso $\varphi^\bullet: \mathrm{Cone}(i^\bullet) \to X^\bullet[1]$ in $K(\mathcal{A})$

s.t.

$$\begin{tikzcd} Y^\bullet \arrow[r, "i^\bullet"] \arrow[d, equal] & \mathrm{Cone}(f^\bullet) \arrow[r, "\binom{1}{0}"] \arrow[d, equal] \arrow[dr, phantom, "(1)"] & \mathrm{Cone}(i^\bullet) \arrow[r, "(0{,}1)"] \arrow[d, "\varphi^\bullet"] \arrow[dr, phantom, "(2)"] & Y^\bullet[1] \arrow[d, equal] \\ Y^\bullet \arrow[r, "i^\bullet"] & \mathrm{Cone}(f^\bullet) \arrow[r, "\pi^\bullet"] & X^\bullet[1] \arrow[r, "-f[1]"] & Y^\bullet[1] \end{tikzcd}$$

commutes.

$$\mathrm{Cone}(i^\bullet)^n = \mathrm{Cone}(f^\bullet)^n \oplus Y^{n+1} = Y^n \oplus X^{n+1} \oplus Y^{n+1}$$

$$d^n_{\mathrm{Cone}(i^\bullet)} = \begin{pmatrix} d_{\mathrm{Cone}(f^\bullet)} & i^{n+1} \\ 0 & -d_Y^{n+1} \end{pmatrix} = \begin{pmatrix} d_Y^n & f^{n+1} & 1 \\ 0 & -d_X^{n+1} & 0 \\ 0 & 0 & -d_Y^{n+1} \end{pmatrix}$$

$\varphi^\bullet : \mathrm{Cone}(i^\bullet) \longrightarrow X^\bullet[1]$ given by

$\varphi^n = (0, 1, 0) : Y^n \oplus X^{n+1} \oplus Y^{n+1} \longrightarrow X^{n+1}$

Define $\psi^\bullet : X^\bullet[1] \longrightarrow \mathrm{Cone}(i^\bullet)$

$\psi^n = \begin{pmatrix} 0 \\ 1 \\ -f^{n+1} \end{pmatrix} : X^{n+1} \longrightarrow Y^n \oplus X^{n+1} \oplus Y^{n+1}$

Check: (i) $\varphi^\bullet$ and $\psi^\bullet$ are morphism of complexes

(ii) The square

$$\begin{tikzcd} \mathrm{Cone}(f^\bullet) \arrow[r, "\binom{1}{0}"] \arrow[d, equal] & \mathrm{Cone}(i^\bullet) \arrow[d, "\varphi^\bullet"] \\ \mathrm{Cone}(f^\bullet) \arrow[r, "\pi^\bullet"] & X^\bullet[1] \end{tikzcd}$$

commutes in $Ch(\mathcal{A})$

(iii) The square

$$\begin{tikzcd} \mathrm{Cone}(i^\bullet) \arrow[r, "{(0,1)}"] & Y^\bullet[1] \\ X^\bullet[1] \arrow[u, "\psi^\bullet"] \arrow[r, "-f^\bullet[1]"] & Y^\bullet[1] \arrow[u, "="'] \end{tikzcd}$$

commutes in $Ch(\mathcal{A})$

(iv): $\varphi^\bullet \circ \psi^\bullet = 1_{X^\bullet[1]}$.

(v) We show $1 - \psi^\bullet \circ \varphi^\bullet$ is nullhomotopic
($\Rightarrow \psi^\bullet, \varphi^\bullet$ mutual inverse iso's in $K(\mathcal{A})$)

$$1-\psi^n\circ\varphi^n=\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}-\begin{pmatrix}0\\1\\-f^{n+1}\end{pmatrix}\circ(0,1,0)$$

$$=\begin{pmatrix}1&0&0\\0&0&0\\0&f^{n+1}&1\end{pmatrix}$$

set $h^n=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}:\underset{Y^n\oplus X^{n+1}\oplus Y^{n+1}}{\underset{\parallel}{\mathrm{Cone}(i^\bullet)^n}}\longrightarrow\underset{Y^{n-1}\oplus X^{n}\oplus Y^{n}}{\underset{\parallel}{\mathrm{Cone}(i^\bullet)^{n-1}}}$

Then

$$d^{n-1}_{\mathrm{Cone}(i^\bullet)}\circ h^n+h^{n+1}\circ d^n_{\mathrm{Cone}(i^\bullet)}$$

$$=\begin{pmatrix}d_Y^{n-1}&f^n&1\\0&-d_X^n&0\\0&0&-d_Y^n\end{pmatrix}\circ\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}+\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}\circ\begin{pmatrix}d_Y^n&f^{n+1}&1\\0&-d_X^{n+1}&0\\0&0&-d_Y^{n+1}\end{pmatrix}$$

$$=\begin{pmatrix}1&0&0\\0&0&0\\-d_Y^n&0&0\end{pmatrix}+\begin{pmatrix}0&0&0\\0&0&0\\d_Y^n&f^{n+1}&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&0&0\\0&f^{n+1}&1\end{pmatrix}$$

$$=1-\psi^n\circ\varphi^n.$$

(iv) + (v) $\Rightarrow\psi^\bullet,\varphi^\bullet$ mutual inverse iso's in $K(\mathcal{A})$.

(iii) $\Rightarrow$
$$\begin{tikzcd} \mathrm{Cone}(i^\bullet) \arrow[r, "{(0,1)}"] \arrow[d, "\varphi^\bullet"] & Y^\bullet[1] \arrow[d, "="] \\ X^\bullet[1] \arrow[r, "-f^\bullet[1]"] & Y^\bullet[1] \end{tikzcd}$$
commutes in $K(\mathcal{A})$. ∎

<u>Lemma</u>: $(K(\mathcal{A}), [1], \Delta)$ satisfies (T1)

<u>Pf</u>: Need to show

- Δ closed under iso: By definition ✓
- $\forall f: X^\bullet \to Y^\bullet$ in $K(\mathcal{A})$ exists $(X^\bullet \xrightarrow{f} Y^\bullet \to Z^\bullet \to X^\bullet[1]) \in \Delta$:

  choose a lift $f^\bullet: X^\bullet \to Y^\bullet$ to $Ch(\mathcal{A})$ and a standard triangle
  $$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \to \mathrm{Cone}(f^\bullet) \to X^\bullet[1] \checkmark$$
- $(X^\bullet \xrightarrow{1_X} X^\bullet \to 0 \to X^\bullet[1]) \in \Delta$:

Since $\mathrm{Cone}(0 \to X^\bullet) = X^\bullet$, have standard triangle
$$(0 \to X^\bullet \xrightarrow{1_X} X^\bullet \to 0) \in \Delta$$
Claim follow now from (T2) ∎