Lecture 4

Equivalences of categories

<u>Def</u>: A functor $F: \mathcal{C} \to \mathcal{D}$ is an <u>equivalence</u> if $\exists\, G: \mathcal{D} \to \mathcal{C}$ such $F \circ G$ is nat iso to $Id_{\mathcal{D}}$ and $G \circ F$ is nat iso to $Id_{\mathcal{C}}$.

G + <u>quasi inverse</u> of F

<u>Idea</u>: We want to say when to categories are "the same"

Isomorphism of categories is too strong

Equivalence of categories is the correct notion

<u>Theorem</u> $F: \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if it is full, faithful and dense.

<u>Proof</u>: Assume F is an equivalence

$G: \mathcal{D} \to \mathcal{C}$ quasi-inverse.

$\alpha: GF \xrightarrow{\cong} Id_{\mathcal{C}}$ $\quad \beta: FG \xrightarrow{\cong} Id_{\mathcal{D}}$.

nat iso's.

<u>F dense</u>: Let $D \in \mathcal{D}$ Then

$\beta_D: F(G(D)) \xrightarrow{\cong} D$ iso

F(object in $\mathcal{C}$)

F faithful: Let $f_1, f_2 : X \to Y$ in $\mathcal{C}$,

Assume $F(f_1) = F(f_2)$. Have commutative squares

$$\begin{tikzcd} GF(X) \arrow[r, "GF(f_1)"] \arrow[d, "\alpha_X"'] & GF(Y) \arrow[d, "\alpha_Y"] \\ X \arrow[r, "f_1"'] & Y \end{tikzcd} \qquad \begin{tikzcd} GF(X) \arrow[r, "GF(f_2)"] \arrow[d, "\alpha_X"] & GF(Y) \arrow[d, "\alpha_Y"] \\ X \arrow[r, "f_2"'] & Y \end{tikzcd}$$

α_X, α_Y isomorphisms

$$\Rightarrow f_1 = \alpha_Y \circ GF(f_1) \circ \alpha_X^{-1} = \alpha_Y \circ GF(f_2) \circ \alpha_X^{-1} = f_2$$

$(GF(f_1) = GF(f_2))$

F full: $g : F(X) \to F(Y)$ morphism in $\mathcal{D}$.

Consider

$$\begin{tikzcd} GF(X) \arrow[r, "G(g)"] \arrow[d, "\alpha_X"'] & GF(Y) \arrow[d, "\alpha_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \quad (*)$$

$f := \alpha_Y G(g) \alpha_X^{-1}$

Then $F(f), g \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$

Want to show they are equal.

Consider

$$\begin{tikzcd} GF(X) \arrow[r, "GF(f)"] \arrow[d, "\alpha_X"'] & GF(Y) \arrow[d, "\alpha_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commute by naturality $\Rightarrow GF(f) = \alpha_Y^{-1} f \alpha_X$

From $(*)$, $G(g) = \alpha_Y^{-1} f \alpha_X$

$\Rightarrow GF(f) = G(g)$

Now since $G: \mathcal{D} \to \mathcal{C}$ is an equivalence (why?), we know from the argument above that G is faithful

Hence $GF(f) = G(g) \Rightarrow F(f) = g$

so F is full.

Converse: Assume $F: \mathcal{C} \to \mathcal{D}$ is full, faithful and dense.

For each $Z \in \mathcal{D}$, choose (!) $A_Z \in \mathcal{C}$ and an iso $\Theta_Z : F(A_Z) \xrightarrow{\cong} Z$. ($F$ dense)

$\forall\, Z \xrightarrow{g} W$ in $\mathcal{D}$, consider

$$F(A_Z) \xrightarrow{\Theta_Z} Z \xrightarrow{g} W \xrightarrow{\Theta_W^{-1}} F(A_W)$$

F full & faithful $\Rightarrow \exists$ unique map

$f_g : A_Z \to A_W$ in $\mathcal{C}$ s.t.

$F(f_g) = \Theta_W^{-1} \circ g \circ \Theta_Z$. Define $G : \mathcal{D} \to \mathcal{C}$ by

$G(Z) = A_Z \quad \forall\, Z \in \mathcal{D}$

$G(g) = G(f_g) \quad \forall$ morphisms g in $\mathcal{D}$.

Then:

- G is a functor (check!)
- $\Theta = \{\Theta_Z : FG(Z) \to Z \mid Z \in \mathcal{D}\}$ is a nat iso $F \circ G \to \mathrm{Id}_{\mathcal{D}}$ (check!)

$\forall\, X \in \mathcal{C}$, consider the morphism

$\Theta_{F(X)} : FGF(X) \longrightarrow F(X)$

F full & faithful $\Rightarrow \exists$ unique morphism

$\xi_X : GF(X) \longrightarrow X$ s.t. $G(\xi_X) = \theta_{F(X)}$.

Then $\xi = \{\xi_X : GF(X) \to X \mid X \in \mathcal{C}\}$ is a nat iso $GF \xrightarrow{\simeq} Id_{\mathcal{C}}$ (check!)

$\blacksquare$

<u>Example</u>: $\mathbb{F}$ field.

$Mat_{\mathbb{F}}$ category:

$Obj\ Mat_{\mathbb{F}} = \mathbb{N} \cup \{0\}$

$Hom_{Mat_{\mathbb{F}}}(m,n) = \{n \times m$ matrices over $\mathbb{F}\}$

composition = matrix multiplication

have natural functor

$$Mat_{\mathbb{F}} \longrightarrow \text{mod } \mathbb{F}$$
$$n \longrightarrow \mathbb{F}^n$$

This is full, faithful and dense hence it is an equivalence!

Adjoint functors

Def: $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ functors
(F, G) is an adjoint pair if
$\forall X \in \mathcal{C}$ $Y \in \mathcal{D}$ we have a natural isomophism

$$\phi_{XY}: \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(X, G(Y)).$$

Here natural means that for all $f: X \to X'$ in $\mathcal{C}$ & $g: Y \to Y'$ in $\mathcal{D}$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(X'), Y) \arrow[r, "\mathrm{Hom}_{\mathcal{D}}(F(f), Y)"] \arrow[d, "\phi_{X', Y}"] & \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \arrow[r, "\mathrm{Hom}_{\mathcal{D}}(F(X), g)"] \arrow[d, "\phi_{X, Y}"] & \mathrm{Hom}_{\mathcal{D}}(F(X), Y') \arrow[d, "\phi_{X, Y'}"] \\
\mathrm{Hom}_{\mathcal{C}}(X', G(Y)) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(f, G(Y))"] & \mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(X, G(g))"] & \mathrm{Hom}_{\mathcal{C}}(X, G(Y'))
\end{tikzcd}$$

commutes.

Remark $\cdot$ $\mathcal{C}, \mathcal{D}$ category.
Have category $\mathcal{C} \times \mathcal{D}$ given by
$\mathrm{Obj}(\mathcal{C} \times \mathcal{D}) = \mathrm{Obj}\, \mathcal{C} \times \mathrm{Obj}\, \mathcal{D} = \{(X, Y) \mid X \in \mathcal{C}, Y \in \mathcal{D}\}$

$\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((X,Y),(X',Y'))$

$= \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{D}}(X',Y')$.

Composition defined componentwise.

- $\mathcal{C}$ cat. The association
$(X,Y) \longmapsto \mathrm{Hom}_{\mathcal{C}}(X,Y)$
can be made into a functor
$\mathrm{Hom}_{\mathcal{C}}(-,\cdot) : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Set}$ (check!)

- A pair (F,G), $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ is adjoint
$\iff$ there exists a natural isomorphism
$$\mathrm{Hom}_{\mathcal{D}}(F(-),\cdot) \to \mathrm{Hom}_{\mathcal{C}}(-,G(\cdot))$$
of functors $\mathcal{C}^{op} \times \mathcal{D} \to \mathrm{Set}$

- (F,G) adjoint, say F <u>left adjoint</u> to G, G <u>right adjoint</u> to F

<u>Exercise</u>: $(X,\leq)$, $(Y,\leq)$ posets.
What is an adjoint pair (F,G) when
$F: \mathcal{C}_{(X,\leq)} \to \mathcal{C}_{(Y,\leq)}$, $G: \mathcal{C}_{(Y,\leq)} \to \mathcal{C}_{(X,\leq)}$?