Lemma: $(K(\mathcal{A}), [1], \Delta)$ satisfies (T3)

Pf: Need to show dashed arrow exists

$$\begin{tikzcd} X_1^\bullet \arrow[r] \arrow[d, "a^\bullet"] & Y_1^\bullet \arrow[r] \arrow[d, "b^\bullet"] & Z_1^\bullet \arrow[r] \arrow[d, dashed] & X_1^\bullet[1] \arrow[d, "a^\bullet[1]"] & \in \Delta \\ X_2^\bullet \arrow[r] & Y_2^\bullet \arrow[r] & Z_2^\bullet \arrow[r] & X_2^\bullet[1] & \in \Delta \end{tikzcd}$$

s.t. diagram commutes. Suffices to show for standard triangles

$$\begin{tikzcd} X^\bullet \arrow[r, "f^\bullet"] \arrow[d, "u^\bullet"] & Y^\bullet \arrow[r] \arrow[d, "v^\bullet"] \arrow[dr, phantom, "(1)"] & \mathrm{Cone}(f^\bullet) \arrow[r] \arrow[d, dashed, "\exists w^\bullet"] \arrow[dr, phantom, "(2)"] & X^\bullet[1] \arrow[d, "u^\bullet[1]"] \\ Z^\bullet \arrow[r, "g^\bullet"] & W^\bullet \arrow[r] & \mathrm{Cone}(g^\bullet) \arrow[r] & Z^\bullet[1] \end{tikzcd}$$

commutes in $K(\mathcal{A})$ = commutes up to homotopy in $Ch(\mathcal{A})$

This means: $\exists\, h^i : X^i \to W^{i-1}$ $\forall i \in \mathbb{Z}$ s.t.

$$v^i \circ f^i - g^i \circ u^i = d_W^{i-1} \circ h^i + h^{i+1} \circ d_X^i$$

define $w^n : \mathrm{Cone}(f^\bullet)^n \longrightarrow \mathrm{Cone}(g^\bullet)^n$

$$\mathrm{Cone}(f^\bullet)^n = Y^n \oplus X^{n+1} \xrightarrow{\begin{pmatrix} v^n & h^{n+1} \\ 0 & u^{n+1} \end{pmatrix}} W^n \oplus Z^{n+1} = \mathrm{Cone}(g^\bullet)^n$$

Check: $w^\bullet$ is a morphism of complexes

- square (1) and (2) commutes ■

Lemma $(K(\mathcal{A}), [1], \Delta)$ satisfies (T4):

Pf: Suffices to show for standard triangles. Consider solid part of diagram

$$
\begin{tikzcd}
X^\bullet \arrow[r, "f^\bullet"] \arrow[d, "="] & Y^\bullet \arrow[r, "\binom{1}{0}"] \arrow[d, "g^\bullet"] & \mathrm{Cone}(f^\bullet) \arrow[r, "(0\ 1)"] \arrow[d, dashed, "\begin{pmatrix} g^\bullet & 0 \\ 0 & 1 \end{pmatrix}"] & X^\bullet[1] \arrow[d, "="] \\
X^\bullet \arrow[r, "g^\bullet \circ f^\bullet"'] & Z^\bullet \arrow[r, "\binom{1}{0}"] \arrow[d] & \mathrm{Cone}(g^\bullet \circ f^\bullet) \arrow[r, "(0\ 1)"] \arrow[d, dashed, "\begin{pmatrix} 1 & 0 \\ 0 & f^\bullet[1] \end{pmatrix}"] & X^\bullet[1] \arrow[d, "f^\bullet[1]"] \\
& \mathrm{Cone}(g^\bullet) \arrow[r, "="] \arrow[d] & \mathrm{Cone}(g^\bullet) \arrow[r, "(0\ 1)"] \arrow[d, dashed, "\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}"] & Y^\bullet[1] \\
& Y^\bullet[1] \arrow[r, "\binom{1}{0}"] & \mathrm{Cone}(f^\bullet)[1] &
\end{tikzcd}
$$

Define dashed arrows as indicated

Remains to check:

- The dashed arrows are morphisms of chain complexes
- every square in the diagram commutes

- the third column is isomorphic in $K(\mathcal{A})$ to a standard triangle.

This is left as an exercise ▪

Next, want to show $\mathrm{Ext}^n_{\mathcal{A}}(A,B)$ iso to morphism spaces in $K(\mathcal{A})$

<u>Lemma</u>: $X \in \mathcal{A}$ $A^\bullet = (\cdots \to A^{-1} \to A^0 \to A^1 \to \cdots)$. Consider

$$\mathrm{Hom}^n_{\mathcal{A}}(A^\bullet, X) = (\cdots \to \mathrm{Hom}_{\mathcal{A}}(A^1, X) \to \mathrm{Hom}_{\mathcal{A}}(A^0, X) \to \mathrm{Hom}_{\mathcal{A}}(A^{-1}, X) \to \cdots$$

Then

$$Z^n \mathrm{Hom}_{\mathcal{A}}(A^\bullet, X) = \mathrm{Hom}_{Ch(\mathcal{A})}(A^\bullet, X[n])$$
$$H^n \mathrm{Hom}_{\mathcal{A}}(A^\bullet, X) = \mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, X[n])$$

<u>Pf</u>: Exercise ▪

$p: \mathcal{A} \to K(\mathcal{A})$ $X \longmapsto pX$ - proj res of X

Have functor $\mathcal{A} \to K(\mathcal{A})$, $A \longmapsto (\cdots \to 0 \to A \to 0 \to \cdots)$
complex concentrated in degree 0

<u>Theorem</u>: $\mathcal{A}$ abelian with enough proj. Then
$$\mathrm{Ext}^n_{\mathcal{A}}(A,B) = \mathrm{Hom}_{K(\mathcal{A})}(pA, B[n])$$

<u>Pf</u>: $\mathrm{Ext}^n_{\mathcal{A}}(A,B) \overset{\text{def}}{=} R^n\mathrm{Hom}(-,B)(A)$
$\overset{\text{def}}{=} H^n \mathrm{Hom}(pA, B[n]) \overset{\text{Lemma}}{\cong} \mathrm{Hom}_{K(\mathcal{A})}(pA, B[n]$