Lecture 12

<u>Corollary</u> The functor

$M \otimes_R - : \mathrm{Mod}\, R \to \mathrm{Mod}\, S$

is right exact.

<u>Pf</u>: Since $M \otimes_R -$ is left adjoint to $\mathrm{Hom}_S(M, -)$, it preserves colimits.
In particular, preserves cokernels, so must be right exact. ∎

<u>Def</u>: M right R-module. M is <u>flat</u> if

$M \otimes_R - : \mathrm{Mod}\, R \to \mathrm{Ab}$ is exact.

Note: Can show any projective R-module is exact
- A flat R-mod is not necessarily projective!

The long exact sequence of homology

Def: $\mathcal{A}$ additive cat

- A (cochain) complex in $\mathcal{A}$ is a sequence

$$\cdots \xrightarrow{d^{-2}} A^{-1} \xrightarrow{d^{-1}} A^{0} \xrightarrow{d^{0}} A^{1} \xrightarrow{d^{1}} \cdots = A^{\bullet}$$

of morphisms in $\mathcal{A}$ s.t $d^{i} \circ d^{i-1} = 0 \ \forall i \in \mathbb{Z}$.

abelian: $B^{n}(A^{\bullet}) := \operatorname{Im} d^{n-1}$ nth boundary of $A^{\bullet}$

$Z^{n}(A^{\bullet}) := \operatorname{Ker} d^{n}$ n-cycles of $A^{\bullet}$

Since $d^{n} \circ d^{n-1} = 0$, have canonical monomorphism $B^{n}(A) \overset{k_n}{\hookrightarrow} Z^{n}(A)$

The nth cohomology of $A^{\bullet}$ is $H^{n}(A^{\bullet}) := \operatorname{Coker} k_n$

(For $\mathcal{A} = \operatorname{Mod} R$, $B^{n}(A)$ submodule of $Z^{n}(A)$ and $H_n(A) = \frac{Z^{n}(A)}{B^{n}(A)}$ factor group)

Ex

Algebraic topology: X topological space

Singular homology $H_n(X)$ obtained by taking homology of a chain complex associated to X, namely the singular complex

$H_n(X)$ invariant of X

__Modules:__ R ring. M, N R-modules

Will see that we can construct chain complex $C_\bullet = C_\bullet^{M,N}$ s.t. $H^n(C_\bullet) = \mathrm{Ext}^n_R(M, N)$, nth Ext-group (see motivation).

Can also define cohomology in the dual way, as the following result shows.

__Lemma__ $\mathcal{A}$ abelian, $A^\bullet$ complex in $\mathcal{A}$. The epi $A^n \to B^{n+1}(A^\bullet)$ factors through $A^n \twoheadrightarrow \operatorname{Coker} d^{n-1}$, and

$$H^n(A^\bullet) \cong \operatorname{Ker}(\operatorname{Coker} d^{n-1} \twoheadrightarrow B^{n+1}(A))$$

__Pf:__ Have commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & B^n(A) \arrow[r, hook] \arrow[d, "h_n"] & A^n \arrow[r] \arrow[d, equal] & \operatorname{Coker} d^{n-1} \arrow[r] \arrow[d, dashed, "l_n"] & 0 \\
0 \arrow[r] & Z^n(A) \arrow[r, hook] & A^n \arrow[r] & B^{n+1}(A) \arrow[r] & 0
\end{tikzcd}$$

right vertical map exists by commutativity of left square. By Snake lemma have exact sequence

$$0 \to \operatorname{Ker} \ell_n \to \operatorname{Coker} k_n \to 0$$

Hence $\operatorname{Ker} \ell_n \cong \operatorname{Coker} k_n = H^n(A^\bullet)$

<u>Def</u>: $\mathcal{A}$ additive
$C(\mathcal{A})$ category of complexes in $\mathcal{A}$,
where

$$\operatorname{Hom}_{C(\mathcal{A})}(A^\bullet, B^\bullet) = \left\{ (f^i)_{i \in \mathbb{Z}} \;\middle|\; \begin{array}{l} f^i : A^i \to B^i \text{ s.t.} \\ f^i \circ d^{i-1}_{A^\bullet} = d^{i-1}_{B^\bullet} \circ f^{i-1} \end{array} \right\}$$

i.e. comm diagrams

$$\begin{tikzcd}
\arrow[r] & A^{-1} \arrow[r, "d^{-1}_{A^\bullet}"] \arrow[d, "f^{-1}"'] & A^0 \arrow[r, "d^0_{A^\bullet}"] \arrow[d, "f^0"] & A^1 \arrow[r, "d^1_{A^\bullet}"] \arrow[d, "f^1"] & \cdots \\
\arrow[r] & B^{-1} \arrow[r, "d^{-1}_{B^\bullet}"] & B^0 \arrow[r, "d^0_{B^\bullet}"] & B^1 \arrow[r, "d^1_{B^\bullet}"] & {}
\end{tikzcd}$$

ote $\mathcal{A}$ abelian • $C(\mathcal{A})$ abelian, kernels & cokernels taken componentwise.

• $A^\bullet \mapsto Z^n(A^\bullet)$ $\quad A^\bullet \mapsto B^n(A^\bullet)$
and $A^\bullet \mapsto H^n(A^\bullet)$ induce additive functors $C(\mathcal{A}) \to \mathcal{A}$

- For $f^\bullet : A^\bullet \to B^\bullet$ in Mod R
- $Z^n(f)(a) = f^n(a)$ for $a \in Z^n(A^\bullet) \subset A^n$
- $B^n(f)(a) = f^n(a)$ for $a \in \operatorname{Im} d_A^{n-1} \subset A^n$
- $H^n(f)(a + \operatorname{im} d_A^{n-1}) = f^n(a) + \operatorname{im} d_B^n$ for $a \in Z^n(A) \subset A^n$

__Theorem__ (Long exact sequence of homology)

$0 \to A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \to 0$ exact sequence in $C(\mathcal{A})$

Then have long exact sequence

$$\cdots \to H^n(A^\bullet) \xrightarrow{H^n(f^\bullet)} H^n(B^\bullet) \xrightarrow{H^n(g^\bullet)} H^n(C^\bullet) \xrightarrow{\partial} H^{n+1}(A^\bullet) \xrightarrow{H^{n+1}(f^\bullet)} H^{n+1}(B) \to$$

__Pf__: Consider

$$\begin{tikzcd}
0 \arrow[r] & A^i \arrow[r, "f^i"] \arrow[d, "d_A^i"] & B^i \arrow[r, "g^i"] \arrow[d, "d_B^i"] & C^i \arrow[r] \arrow[d, "d_C^i"] & 0 \\
0 \arrow[r] & A^{i+1} \arrow[r, "f^{i+1}"] & B^{i+1} \arrow[r] & C^{i+1} \arrow[r] & 0
\end{tikzcd}$$

Taking kernels is left exact and taking cokernels is right exact (follows from Snake lemma or Problem 7 d) on exercise sheet 3)

Hence get comm diagram with exact rows

$$\begin{tikzcd}
& \operatorname{Coker} d_A^{n-1} \arrow[r] \arrow[d] & \operatorname{Coker} d_B^{n-1} \arrow[r] \arrow[d] & \operatorname{Coker} d_C^{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z^{n+1}(A) \arrow[r] & Z^{n+1}(B) \arrow[r] & Z^{n+1}(C) &
\end{tikzcd}$$

where the vertical maps are

$$\mathrm{Coker}\, d_A^{n-1} \twoheadrightarrow B^{n+1}(A) \hookrightarrow Z^{n+1}(A)$$

and similar for $B^\bullet$, $C^\bullet$.

Thus by Snake lemma get exact sequence

$$\begin{tikzcd} H^n(A^\bullet) \arrow[r, "H^n(f^\bullet)"] & H^n(B^\bullet) \arrow[r, "H^n(g^\bullet)"] & H^n(C^\bullet) \arrow[dll, "\partial"'] \\ H^{n+1}(A^\bullet) \arrow[r, "H^{n+1}(f^\bullet)"] & H^{n+1}(B^\bullet) \arrow[r, "H^{n+1}(g^\bullet)"] & H^{n+1}(C^\bullet) \end{tikzcd}$$

repeating argument $\forall n \in \mathbb{Z}$ proves the claim ■

Note: $C^\bullet$ complex, then $C^\bullet$ exact $\Leftrightarrow H^n(C^\bullet) = 0 \ \forall n \in \mathbb{Z}$.

Get following corollary

<u>Corollary</u> $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ exact sequence in $C(\mathcal{A})$. If two out of $A^\bullet$, $B^\bullet$, $C^\bullet$ are exact, then all of $A^\bullet$, $B^\bullet$, $C^\bullet$ are exact.

<u>Cones and quasi-isomorphisms</u>

We often only care about chain complexes up to its homology.

<u>Def</u>: $f^\bullet: A^\bullet \to B^\bullet$ morphism of complexes.
$f^\bullet$ is a <u>quasi-isomorphism</u> if and only if
$H^n(f^\bullet)$ is an isomorphism $\forall n \in \mathbb{Z}$.

How to detect if $f^\bullet$ is a quasi-iso?

- If $f^\bullet$ monomorphism, then get ses of cplxs

$$0 \to A^\bullet \xrightarrow{f^\bullet} B^\bullet \to \operatorname{Coker} f^\bullet \to 0$$

$\rightsquigarrow$ get long exact sequence in homology.
Hence $f^\bullet$ quasi-iso $\iff H^n(f^\bullet)$ iso $\forall n \in \mathbb{Z}$

$\iff H^n(\operatorname{Coker} f^\bullet) \cong 0 \ \forall n \in \mathbb{Z}$

$\iff \operatorname{Coker} f^\bullet$ exact.

What if $f^\bullet$ not mono, can we do something similar?

Yes! Using the cone construction.

<u>Def</u> (Cone): Let $f^\bullet: A^\bullet \to B^\bullet$ morphism in $Ch(\mathcal{A})$.
The <u>cone of $f^\bullet$</u>, denoted $\operatorname{Cone}(f^\bullet)$,
is the complex